Solutions of differential-algebraic equations as outputs of LTI systems: application to LQ control problems

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Abstract

In this paper we synthesize behavioral ideas with geometric control theory and propose a unified geometric framework for representing all solutions of a Linear Time Invariant Differential-Algebraic Equation (DAE-LTI) as outputs of classical Linear Time Invariant systems (ODE-LTI). An algorithm for computing an ODE-LTI that generates solutions of a given DAE-LTI is described. It is shown that two different ODE-LTIs which represent the same DAE-LTI are feedback equivalent. The proposed framework is then used to solve an LQ optimal control problem for DAE-LTIs with rectangular matrices.

1 Introduction

Consider a linear time invariant differential-algebraic equation (abbreviated by DAE-LTI) of the form

\[
\frac{d(Ex(t))}{dt} = Ax(t) + Bu(t)
\]  

(1)

with arbitrary rectangular matrices \( E, A \in \mathbb{R}^{c \times n} \) and \( B \in \mathbb{R}^{c \times m} \). In this paper we discuss how to represent solutions of (1) as outputs of linear time invariant ordinary differential equations (abbreviated by ODE-LTI). This representation is then applied to derive necessary and sufficient solvability conditions for LQ optimal control problems with DAE-LTI constraints.

Non-regular DAE-LTIs in the form (1) arise in control from several sources. They could either be a result of modeling physical systems, or arise as a result of interconnecting several (possibly regular) DAEs. Indeed, regular DAE-LTIs are not closed under interconnection and so by applying a state-feedback to a regular DAE-LTIs one may arrive at a non-regular DAE-LTI [26]. Another nice example of non-regular DAE-LTIs are ODE-LTI with unknown external inputs. For instance, such systems arise when approximating Partial Differential Equations (PDEs) by ODEs. Then the approximation error can be viewed as an unknown input [35,36]. Such systems can be modelled by DAE-LTIs if the inputs are viewed as a part of the state. LQ optimal control for DAE-LTIs in the form (1) was studied by many authors [3,5,13-15,22,24,25,33]. In the mentioned papers solutions of (1) were defined either as smooth functions or as distributions. In many applications, however, the solution of (1) cannot be assumed to be smooth or to be a distribution, the former being too restrictive, while the latter does not correspond to the physical meaning of the DAE-LTI state. The present paper is motivated by the need for a framework which (A) provides a simple description of all solutions of DAE (1), for which \( Ex \) is absolutely continuous and \( x,u \) are locally integrable, and (B) allows to efficiently compute solutions of LQ optimal control problems for such DAE-LTIs. As an example of an application which requires such a framework, we mention the problem of state estimation for DAE-LTIs which arises in numerical analysis.

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Many linear PDEs can be viewed as a linear time-invariant system \( u_t = Au, \ u(0) = u_0 \in V \) with an infinite dimensional state space \( V \) (for instance, Sobolev spaces of weakly differentiable functions). The precise choice of \( V \) depends on the type of a differential operator \( A \). Often, it is possible to find a suitable system of orthogonal basis vectors in \( V \) and identify \( u \) with the infinite vector of its coordinates w.r.t. to this basis. In order to compute the solution \( u \), the infinite vector \( u = (u_1, u_2, \ldots, u_k, \ldots)^T \) is approximated by its truncation \( u^h = P^h u = (u_1, u_2, \ldots, u_k)^T \). In many applications, part of the state \( u \) can be measured experimentally, i.e. finite dimensional measurement vectors \( y(t) = Cu(t) \) are available, where \( C \) is an “observation operator”. In [35, 36] it was shown that the truncated vector \( z := u^h \) satisfies a DAE-LTI in the following form:

\[
\begin{align*}
\frac{d(Fz(t))}{dt} &= Gz(t) + f, \\
y(t) &= Hz(t) + \eta(t),
\end{align*}
\]

(2)

where \( F, G, H \) are certain rectangular matrices representing truncations of \( A \) and \( C \), and \( f, \eta \) represent the terms which model the effect of the truncation error. Note that the time derivative of \( z = u^h \) is a function of all the components of \( u = (u_1, \ldots, u_k, \ldots)^T \), not just the first \( h \) ones, hence the error terms \( f \) and \( \eta \). In [35, 36] it was shown that for certain classes of PDEs and certain choice of basis functions, \((z(0), f, \eta)\) belong to the set

\[
\mathcal{E} = \{(z_0, f, \eta) \mid \rho(Fz_0, f, \eta) \leq 1\},
\]

\[
\rho(q_0, f, \eta) := q_0^T \hat{Q}_0 q_0 + \int_0^\infty (f^T Q^{-1} f + \eta^T R^{-1} \eta) dt.
\]

for suitable positive definite matrices \( \hat{Q}_0, Q, R \). That is, [35, 36] proposes a Galerkin-style method for solving PDEs, but unlike the classical methods, it takes into account the truncation error explicitly. The problem is that the obtained equation (2) cannot be solved numerically, since \( f \) and \( \eta \) are not known. However, one could use the experimental data \( y(t) \) to estimate \( z(t) \). Since we have bounds on the norms of \( \eta, f \) and \( z(0) \), we could use a minimax observer to find an estimate of \( z(t) \) such that the maximal (worst-case) difference between this estimate and \( z(t) \) is minimal. From [37, 38] it follows that in order to construct such an observer, we have to solve an LQ control problem, that is to minimize \( \rho \) over solutions of a dual DAE-LTI given by (1) with \( A = -G^T, B = H^T, E = F^T \).

Notice that according to [35, 36] the state \( z \) of (2) is absolutely continuous as it models \( u^h \), and \( F \) is a rectangular matrix. Hence, (i) the dual DAE-LTI will not be regular and may have several solutions (or none at all) from any initial state and any input (see Example 1), and (ii) each solution of the dual DAE-LTI will have an absolutely continuous part and a measurable part. Thus, the usual assumption on regularity, impulse controllability, etc. do not hold for the dual system. Moreover, the solutions cannot be assumed either to be smooth or to be a distribution. These observations clearly indicate that a framework satisfying conditions (A) and (B) is required to estimate the state of (2) and obtain a robust approximation of PDE’s solution.

The aim of the present paper is to propose a framework featuring (A) and (B) for general DAEs-LTI. To this end we use behavioral approach [32] and geometric control theory [27]: namely, given DAE-LTI in the form (1) we achieve

\[
\begin{align*}
\dot{p} &= Ap + B_tv \\
w &= Ctp + D_tv
\end{align*}
\]

(3)

such that the external behavior (set of output trajectories) of (3) coincides with the set of solutions of the given DAE-LTI, and (3) satisfies a number of nice technical conditions detailed in the following section.

Representing solutions of DAE-LTIs as outputs of ODE-LTIs is a classical idea. The earliest method relies on Kronecker canonical form [5,10]. However, the method requires to differentiate the inputs and so either the input is assumed to be smooth or the solution is viewed as a distribution. We take another approach which is based on the observation that solutions of a DAE-LTI can be viewed as output nulling solutions of a suitable ODE-LTI. Various versions of this approach appeared in [1,22,25,33,34]. However, in the cited papers, ODE-LTIs played a role of an auxiliary tool, and hence the constructions were not general, but rather problem specific and existence conditions were tailored to meet the requirements of the problem at hand. In contrast, this paper describes the entire class of associated ODE-LTIs which have a simple system-theoretic interpretation: they are feedback equivalent to a minimal ODE-LTI realizations in the sense of [32] of the solution set of the DAE-LTI at hand. The results [1,33,34]
are special cases of the ones which are presented here, provided the corresponding assumptions are used. The construction of [22, 25] is closely related, but it is not formally a special case due to the different solution concept used in the paper.

The concept of ODE-LTIs associated with DAE-LTIs allows us to easily achieve point (B) above, namely solve the infinite horizon LQ control problem for DAE-LTIs, by reducing it to the classical LQ control problem for ODE-LTIs. In particular, we derive new necessary and sufficient solvability conditions for the infinite horizon LQ control problem in terms of behavioral stabilizability of DAE-LTIs. Specifically, we show that the optimal value of the quadratic cost function is given by a norm of the initial condition which is induced by a unique solution of the algebraic Riccati equation. Moreover, if \( (x^*, u^*) \) is the optimal trajectory verifying \( Ex^*(0) = Ex_0 \), then \( u^* = Kx^* \) for some matrix \( K \). i.e. the optimal input has the form of a feedback. Note that this does not imply that all solutions of the closed-loop system
\[
\frac{d(Ex)}{dt} = (A + BK)x, \ E(x(0)) = Ex_0
\]
are optimal, as the latter may have several solutions, including ones which render the cost function infinite. However, we show that there exist matrices \( K_1, K_2 \) such that \( (x^*, u^*) \) is the only solution of the DAE-LTI
\[
\frac{d(Ex)}{dt} = Ax + Bu, K_1x + K_2u = 0 \text{ which satisfies } E(x(0)) = Ex_0.
\]
The additional algebraic constraint \( K_1x + K_2u = 0 \) can be thought of as a generalization of state-feedback concept and can be interpreted as a controller in the sense of behavioral approach [32] (see Remark 1). Note that controllers which are not of feedback form can still be implemented and in fact are widely used for controlling physical devices [30]. Moreover, for the purposes of observer design [37, 38] it is sufficient that at least one trajectory of the closed-loop system is optimal.

The literature on optimal control for DAE-LTIs is vast. For an overview we refer to [18] and the references therein. To the best of our knowledge, the most relevant references on LQ control are [2, 13–15, 22, 24, 25, 28, 33]. In [2, 28] only regular DAEs were considered. The infinite horizon LQ control problem for non-regular DAE was also addressed in [33], however there it is assumed that the DAE has a solution from any initial state. We consider existence of a solution from a particular initial condition, as opposed to [22, 24, 25]. This is done both for the sake of generality and in order to address the requirements of already mentioned observer design problems [37, 38]. Furthermore, in contrast to [22–25], where only sufficient conditions are presented, in this paper we present necessary and sufficient conditions. Moreover, the cost function considered here differs from the one of [22–25], as it includes a terminal cost term \( x^T(t_1)E^TQ_0Ex(t_1) \). Note that the latter term is indispensable to transform an observer design problem into a dual control problem (see [38, Theorem 1] for the further details). In addition, we allow non-smooth solutions. This leads to subtle but important technical differences.

We note that in [7] the behavioral approach was used for LQ control of DAE-LTIs, however, the LQ problem considered in [7] is different from the one of this paper and it does not present detailed algorithms. The results of [13–16] provide sufficient conditions for existence of an optimal controller for stationary DAEs; these conditions involve existence of a solution to an algebraic Riccati equation. In contrast, we provide conditions which are necessary and sufficient, and are, therefore, less restrictive. To illustrate this we describe an LQ control problem for a simple DAE-LTI such that the conditions of [13–16] are not satisfied (see discussion after Example 1). This LQ control problem arises as a dual of an observer design problem for a DAE-LTI of the form (2). On the other hand, this generality comes at price: the sufficient conditions of [13–16] yield a feedback such that all trajectories of the closed-loop system are optimal. In contrast, the solution of this paper does not always yield such a feedback law.

An extended version of this paper is available at [19] and its preliminary version appeared in [37]. With respect to [37] the main difference is that we included detailed proofs, and provided necessary and sufficient conditions for existence of a solution for the infinite horizon optimal control problem. The solution of the finite horizon optimal control problem was already presented in [34]. In contrast to [34] we consider the infinite horizon case too, and the algorithm of [34] for computing an ODE-LTI that generates solutions of a given DAE-LTI is one of many possible implementations of the generic procedure of this paper.

**Outline of the paper** In Section 2 we present the notion of an ODE-LTI associated with a DAE-LTI and prove that all ODE-LTI representing the same DAE-LTI are feed-back equivalent. In Section 3 we apply this result to solve the infinite horizon LQ control problem. In Section 4 we present a numerical example.

**Notation** \( I_n \) denotes the \( n \times n \) identity matrix; for an \( n \times n \) matrix \( S \), \( S > 0 \) means \( x^TSx > 0 \) for all \( x \neq 0 \in \mathbb{R}^n \), \( F^+ \) denotes the Moore-Penrose pseudoinverse of the matrix \( F \). Consider an interval \( I \subseteq \mathbb{R} \) of the form \( [a, b], [a, +\infty) \), \( (-\infty, a], a, b \in \mathbb{R} \), or \( I = \mathbb{R} \). For an integer \( p > 0 \) denote by \( L^p(I, \mathbb{R}^n) \) (or simply by \( L^p(I) \)) the space of all measurable functions \( f : I \to \mathbb{R}^n \) such that \( \int_I ||f||^pdm < +\infty \), where \( m \) is the Lebesgue measure on \( \mathbb{R} \) (see [20] for more details). Let \( L^p_{loc}(I, \mathbb{R}^n) = \{ f : I \to \mathbb{R}^n \mid f \in L^p(K, \mathbb{R}^n), \forall K \subseteq I, K \text{ is a compact interval } \} \), i.e. the restriction
of \( f \in L^p_{loc}(I, \mathbb{R}^n) \) onto any compact sub-interval \( K \) of \( I \) is in \( L^p(K, \mathbb{R}^n) \). Note that \( L^p_{loc}(I, \mathbb{R}^n) = L^p(I, \mathbb{R}^n) \) for compact intervals \( I \). We will use the usual conventions to denote integrals with respect to the Lebesgue measure, see [20, page 52, Remark 2.21]. In particular, \( \int_I f dm, \int_I^t f(s) ds \) denote the same integral for \( I = [a, b] \), \( a, b \in \mathbb{R} \) or \( I = [a, +\infty) \). Denote by \( AC(I, \mathbb{R}^n) \) the set of all absolutely continuous functions \( f : I \to \mathbb{R}^n \), see [20] for the definition of absolute continuity. Note that if \( f \in AC(I, \mathbb{R}^n) \), then there exists a function \( g \in L^p_{loc}(I, \mathbb{R}^n) \) such that \( f(t) = f(0) + \int_0^t g(s) ds \), \( \forall t \in I \). In accordance with the convention, [20], we say that an equation \( f(t) = f_1(t) + f_2(t) \) holds almost everywhere (write \( f_1(t) = f_2(t) \) a.e. or simply \( f_1 = f_2 \) a.e.) for any measurable functions \( f_1, f_2 : I \to \mathbb{R} \), if there exists a set \( S \subseteq I \), such that \( S \) is of Lebesgue measure zero and for any \( t \in I \), \( t \notin S \), \( f_1(t) = f_2(t) \). Finally, \( f|_A \) stands for the restriction of a function \( f \) onto a set \( A \).

2 Linear systems associated with DAEs

Consider a linear time-invariant differential-algebraic system (DAE-LTI)

\[
\frac{dEx(t)}{dt} = Ax(t) + Bu(t), \tag{4}
\]

Here \( A, E \in \mathbb{R}^{c \times n} \), \( B \in \mathbb{R}^{c \times m} \). In this section we will define a class of ODE-LTIs whose output trajectories are the state and input trajectories of (4) and show that these ODE-LTIs exist and they are unique up to feedback equivalence. To this end, we view the set of solutions of (4) as behaviors in the sense of [29, 32], and we view the ODE-LTIs as their state-space representations. We then state a number of consequences of this fact for the solvability theory of (4). The section is organized as follows. In 2.1 we present the main results. In 2.2 we present the proofs of the results.

2.1 Main results

In order to carry out the program outlined above, we start by defining solutions of (4). In this section, by an interval we mean an interval of one of the following forms: \( I = \mathbb{R} \), \( I = [a, +\infty) \), \( I = (-\infty, a] \), \( I = [a, b] \), \( a, b \in \mathbb{R} \).

**Definition 1** Let \( I \) be an interval. A solution of (4) on the interval \( I \) is a tuple \( (x, u) \in L^1_{loc}(I, \mathbb{R}^n) \times L^1_{loc}(I, \mathbb{R}^m) \) such that \( Ex \) is absolutely continuous, and \( \forall t_0 \in I, \forall t \in I, t \geq t_0 : Ex(t) = Ex(t_0) + \int_{t_0}^{t} (Ax(s) + Bu(s)) ds \). Denote by \( B_{f}(E, A, B) \) the set of all solutions \( (x, u) \in L^1_{loc}(I, \mathbb{R}^n) \times L^1_{loc}(I, \mathbb{R}^m) \) of (4) defined on \( I \).

That is, \( (x, u) \in L^1_{loc}(I, \mathbb{R}^n) \times L^1_{loc}(I, \mathbb{R}^m) \) is a solution of (4) on \( I \) if and only if \( Ex \) is absolutely continuous and \( \frac{dEx(t)}{dt} = Ax(t) + Bu(t) \) a.e. Note that solutions may happen to be non-smooth or even discontinuous (except \( Ex \)), so they may contain jumps. Also distributions (as DAE’s solutions) are not allowed. Hence, in our setting the solution of DAE-LTI has no “impulsive parts”. We stress that if we allowed for distributional solutions then DAE-LTI would have solutions with impulsive parts as we do not restrict matrices \( E, A, B \).

Let us now recall few definitions from the behavioral approach [29]. Consider a linear time-invariant system defined by differential equation (referred as ODE-LTI),

\[
\dot{p} = \bar{A}p + G q, \tag{5}
\]

\[
z = \bar{C}p + \bar{D}q,
\]

where \( \bar{A} \in \mathbb{R}^{r \times r} \), \( G \in \mathbb{R}^{r \times s} \), \( \bar{C} \in \mathbb{R}^{p \times r} \), \( \bar{D} \in \mathbb{R}^{p \times s} \). We identify the ODE-LTI (5) with the corresponding tuples \( (\bar{A}, G, \bar{C}, \bar{D}) \) of matrices. Let \( I \) be an interval and \( B \subseteq L^1_{loc}(I, \mathbb{R}^p) \). Following the definition of [29], we say that the ODE-LTI (5) is a realization of \( B \), if (1) for every \( z \in B \) there exist functions \( p \in AC(I, \mathbb{R}^r) \), \( q \in L^1_{loc}(I, \mathbb{R}^s) \) such that \( \dot{p} = \bar{A}p + G q \) a.e., and \( z = \bar{C}p + \bar{D} q \) a.e., and (2) if \( (p, q) \in AC(I, \mathbb{R}^r) \times L^1_{loc}(I, \mathbb{R}^s) \) is such that \( \dot{p} = \bar{A}p + G q \) a.e. then \( z = \bar{C}p + \bar{D} q \) a.e. for some \( z \in B \). That is, if (5) is a realization of \( B \), then any element of \( B \) is an output trajectory of (5), and conversely any output trajectory of (5) belongs to \( B \), possibly after having been modified on a set of measure zero. In the sequel, we are interested in ODE-LTI realizations of \( B_{f}(E, A, B) \). Note that \( B_{f}(E, A, B) \) can naturally be viewed as a subset of \( L^1_{loc}(I, \mathbb{R}^{n+m}) \), so the definition above can be applied. With this terminology we define the notion of a ODE-LTI system associated with a DAE-LTI.
Theorem 3

An ODE-LTI system of the form

\[ \begin{align*}
\dot{v} &= A_l v + B_l g, \\
\nu &= C_l v + D_l g.
\end{align*} \tag{6} \]

is implied by the first row of the DAE-LTI (4), if the following conditions hold:

1. Either \( D_l^T, B_l^T \) is full column rank: \( \text{Rank}D_l = k \).
2. Let \( C_s \) and \( D_s \) be the matrices formed by the first \( n \) rows of \( C_l \) and \( D_l \) respectively. Then \( ED_s = 0, \text{Rank}EC_s = \hat{n} \).
3. \( \mathcal{J} = (A_l, B_l, C_l, D_l) \) is a realization of \( B_l(E, A, B) \) for any interval \( I \).

Notation 1 (\( \mathcal{M} \)) With the notation above, \( \mathcal{M} \) denotes the Moore-Penrose inverse of \( EC_s \). The matrix \( \mathcal{M} \) will be referred to as the state map of \( (A_l, B_l, C_l, D_l) \).

Theorem 1

Let \( (A_l, B_l, C_l, D_l) \) be an ODE-LTI associated with (4). If \( (x, u) \in B_I(E, A, B) \), and we define the function \( v = MEx \) and \( g = D_l^T((x^T, u^T)^T - C_l MEx) \), then \( v \in AC(I, \mathbb{R}^\hat{n}) \), \( g \in L^1_{loc}(I, \mathbb{R}^k) \), and

\[ \dot{v} = A_l v + B_l g \text{ a.e.} \quad (x^T, u^T)^T = C_l v + D_l g \text{ a.e.}. \tag{7} \]

Conversely, for any \( (v, g) \in AC(I, \mathbb{R}^\hat{n}) \times L^1_{loc}(I, \mathbb{R}^k) \) such that \( \dot{v} = A_l v + B_l g \text{ a.e.} \), \( C_l v + D_l g \in B_I(E, A, B) \).

That is, not only the outputs of the associated ODE-LTI correspond to the solutions of the DAE-LTI, but the state trajectory of the DAE-LTI determines the corresponding state trajectory of the associated ODE-LTI.

The question arises if associated ODE-LTIs exist. The answer is affirmative.

Theorem 2 (Existence)

Consider the DAE-LTI system (4). There exists an ODE-LTI system \( \mathcal{J} = (A_l, B_l, C_l, D_l) \) associated with (4).

The proof of Theorem 2 is constructive and it yields an easy to implement algorithm for computing an associated ODE-LTI. The Matlab code of the algorithm is available at http://sites.google.com/site/mihalypetreczky/.

The next question is whether associated ODE-LTIs of the same DAE-LTI are related in any way. In order to answer this question we need the following terminology.

Definition 3 (Feedback equivalence) Two ODE-LTIs \( \mathcal{J}_i = (A_i, B_i, C_i, D_i), i = 1, 2 \) are said to be feedback equivalent, if there exist a matrix \( K \) and two nonsingular square matrices \( U, T \) of suitable dimensions such that \( (T(A_1 + B_1 K)T^{-1}, TB_1U, (C_1 + D_1 K)T^{-1}, D_1 U) = \mathcal{J}_2 \). We will call \( (T, K, U) \) feedback equivalence from \( \mathcal{J}_1 \) to \( \mathcal{J}_2 \).

Theorem 3

Any two ODE-LTI systems associated with the same DAE-LTI (4) are feedback equivalent.

Existence and uniqueness of associated ODE-LTIs allow us to study existence of solutions for DAE-LTIs from a given initial state.

Definition 4 (Consistency set \( V(E, A, B) \)) We will say that a vector \( z \in \mathbb{R}^c \) is differentiably consistent, if there exists a solution \( (x, u) \) of (4) defined on an interval \( I \subseteq \mathbb{R} \) such that \( 0 \in I \) and \( Ex(0) = z \). We denote by \( V(E, A, B) \) the set of all differentiably consistent vectors \( z \in \mathbb{R}^c \).

Corollary 1

Let \( (A_l, B_l, C_l, D_l) \) be an ODE-LTI associated with the DAE-LTI (4), and let \( C_s \) be the matrix formed by the first \( n \) rows of \( C_l \). Then \( V(E, A, B) = \text{im} EC_s \).

Corollary 2

Let \( I \) be any interval such that \( 0 \in I \). If \( z \) is differentiably consistent, there exists a solution \( (x, u) \) of (4) on \( I \), such that \( Ex(0) = z \). Moreover, \( (x, u) \) can be chosen so that \( x, u \) are smooth functions.
In principle, it could happen that there exists a solution \((x, u)\) on the interval \([0, t_1]\) such that \(z = Ex(0)\), but there exist no solution \((x, u)\) with \(z = Ex(0)\) for a larger interval \([0, t_2]\), \(t_2 \geq t_1\). In this case, the subsequent formulation of the finite and infinite horizon control problem would be more involved. Corollary 2 tells us that this can never happen. Corollary 2 also implies that if there exist a solution \((x, u)\) on \(I\) such that \(Ex(0) = Ex_0\), then there exists a solution \((x, u)\) on \(I\) with \(Ex(0) = 0, x_0\) and \(x\) being differentiable, for any interval \(I\) containing 0. Finally, recall from [5] the notion of impulse controllability. Using [5, Corollary 4.3], we can show the following.

**Corollary 3** For any \(x_0 \in \mathbb{R}^n\), \(Ex_0\) is differentially consistent \(\iff\) (4) is impulse controllable \(\iff\) for any matrix \(Z\) such that \(\text{im} Z = \ker E\), \(\text{rank } [E, A, B] = \text{rank } [E, AZ, B]\).

To conclude this section, we would like to discuss the relationship between the results above and existing results. To begin with, existence of an associated ODE-LTI is not that surprising. Note that (4) can be viewed as a kernel representation of \(B_I(E, A, B)\), if one disregards the subtleties related to smoothness of solutions. It is a classical result that behaviors admitting a kernel representation can be represented as outputs of ODE-LTIs [29, 32]. What makes a separate proof of Theorem 2 necessary are the subtle issues related to differentiability of solutions and the additional properties we require for associated ODE-LTIs. In fact, the proof of Theorem 2 bears a close resemblance to [21] which provides an algorithm for computing a state-space realization of a kernel representation of a behavior. However, unlike [21], the proof of Theorem 2 exploits the specific structure of DAE-LTIs and yields existence of ODE-LTI realizations which satisfy Definition 2.

Feedback equivalence of associated ODE-LTIs stems from minimality theory for behaviors. Recall that according to [29] an ODE-LTI \((A, B, C, D)\) is a minimal realization of a behavior \(B \subseteq L_{\text{loc}}^1(I, \mathbb{R}^p)\), if \((A, B, C, D)\) is a realization of \(B\) and for any other ODE-LTI \((A', B', C', D')\) which is a realization of \(B\), the number of state variables of \((A, B, C, D)\) is not greater than the number of state variables of \((A', B', C', D')\). In [29] it was shown that any two minimal-state-space representations of the same behavior are feedback equivalent. It turns out that associated ODE-LTIs are in fact minimal:

**Corollary 4 (Minimality)**

1. Any ODE-LTI system associated with (4) is a minimal state-space representation of the behavior \(B_{\mathbb{R}}(E, A, B)\).
2. Conversely, if \((A, B, C, D)\) is a minimal realization of \(B_{\mathbb{R}}(E, A, B)\) such that \(B^T D^T\) is either full column rank or it is zero and has one column, then \((A, B, C, D)\) is an ODE-LTI associated with (4).

### 2.2 Proofs

**Proof.** [Theorem 1] Let \(C_s, D_s\) be the matrices formed by the first \(n\) rows of \(C_l, D_l\). From Definition 2 it follows that \(ED_s = 0\) and \(EC_s\) is full column rank. Let \((x, u) \in B_I(E, A, B)\). Since \((A_l, B_l, C_l, D_l)\) is a realization of \(B_I(E, A, B)\), it follows that there exist \((v, g) \in AC(I, \mathbb{R}^n) \times L_{\text{loc}}^1(I, \mathbb{R}^k)\) such that (7) holds. It then follows that \(Ex = EC_sv + ED_sg = EC_sv\) a.e., since \(ED_s = 0\). Note that \(Ex\) and \(EC_sv\) are both absolutely continuous, hence \(Ex = EC_sv\) a.e. implies that \(Ex(t) = EC_sv(t)\) for all \(t \in I\). Finally, \((x^T, u^T) = C_lv + D_lg\) a.e. and the fact that \(D_l\) is either zero or it is full column rank, imply that \(g = D_l^+((x^T, u^T)^T - C_lv) = D_l^+((x^T, u^T)^T - C_lMEx)\) a.e.

In order to show the second statement, notice that since \((A_l, B_l, C_l, D_l)\) is a realization of \(B_I(E, A, B)\), there exist \((x, u) \in B_I(E, A, B)\) such that \((x^T, u^T)^T = C_lv + D_lg\) a.e. Let \((\tilde{x}^T, \tilde{u}^T) = C_lv + D_lg\) with \(\tilde{x} \in L_{\text{loc}}^1(I, \mathbb{R}^n)\) and \(\tilde{u} \in L_{\text{loc}}^1(I, \mathbb{R}^m)\). It then follows that \(E\tilde{x} = EC_sv + ED_sg = EC_sv\) is absolutely continuous. Since \(x = \tilde{x}\) a.e. and \(u = \tilde{u}\) a.e., absolute continuity of \(E\tilde{x}\) implies that \((\tilde{x}, \tilde{u}) \in B_I(E, A, B)\). \(\square\)

In order to present the proof of Theorem 2, we recall the following notions from geometric control theory of linear systems. Consider an ODE-LTI of the form (5). Let \(I\) be an interval and let \(t_0 \in I\). Recall from [27, Definition 7.8] the concept of a weakly unobservable subspace of the ODE-LTI (5). I.e., an initial state \(p_0 \in \mathbb{R}^r\) of (5) is weakly unobservable, if there exist \(p \in AC([0, +\infty), \mathbb{R}^r)\) and \(q \in L_{\text{loc}}^1([0, +\infty), \mathbb{R}^s)\) such that \(\dot{p} = A_p + Gq\) a.e., \(p(0) = p_0\), \(0 = C_p + D_q\) a.e. Following [27], let us denote the set of all weakly unobservable states by \(\mathcal{V}\). Recall from [27, Section 7.3], \(\mathcal{V}\) is a vector space and in fact it can be computed. For technical purposes we will need the following easy extension of [27, Theorems 7.10–7.11].

**Theorem 4** Consider the ODE-LTI (5). With the notation above:
(1) \( \mathcal{V} \) is the largest subspace of \( \mathbb{R}^r \) for which there exists a linear map \( \tilde{F} : \mathbb{R}^r \to \mathbb{R}^s \) such that
\[
(\tilde{A} + G\tilde{F})\mathcal{V} \subseteq \mathcal{V} \quad \text{and} \quad (\tilde{C} + D\tilde{F})\mathcal{V} = 0 \tag{8}
\]
(2) Let \( \tilde{F} \) be a map such that (8) holds for \( \mathcal{V} \). Let \( L \in \mathbb{R}^{q \times k} \) for some \( k \) be a matrix such that \( \text{im} L = \ker \tilde{D} \cap G^{-1}(\mathcal{V}) \).

Choose \( L \) so that \( L \) is full column rank if \( \ker \tilde{D} \cap G^{-1}(\mathcal{V}) \neq \{0\} \), or \( L = 0 \in \mathbb{R}^{q \times 1} \) otherwise.

For any interval \( I \), for any \( p \in AC(I, \mathbb{R}^r) \), \( q \in L^1_{loc}(I, \mathbb{R}^s) \),
\[
\tilde{C}p + \tilde{D}q = 0 \quad \text{for} \ t \in I \ a.e.
\]
if and only if \( p(t) \in \mathcal{V} \) for all \( t \in I \), and there exists \( g \in L^1_{loc}(I, \mathbb{R}^k) \) such that:
\[
q(t) = \tilde{F}p(t) + \tilde{L}g(t) \quad \text{for} \ t \in I \ a.e.
\]

**PROOF.** [Proof of Theorem 4] Part 1 is a reformulation of [27, Theorem 7.10]. For \( I = [0, +\infty) \), Part 2 is a restatement of [27, Theorem 7.11]. For all the other intervals \( I \), the proof is similar to [27, Theorem 7.11]. \( \blacksquare \)

**PROOF.** [Proof of Theorem 2] There exist suitable nonsingular matrices \( S \) and \( T \) such that
\[
SET = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \tag{9}
\]
where \( r = \text{Rank} \ E \). Let
\[
SAT = \begin{bmatrix} \tilde{A} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad SB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{10}
\]
be the decomposition of \( E, A, B \) such that \( \tilde{A} \in \mathbb{R}^{r \times r} \) and \( B_1 \in \mathbb{R}^{r \times m} \). Define
\[
G = [A_{12}, B_1], \quad \tilde{D} = [A_{22}, B_2] \quad \text{and} \quad \tilde{C} = A_{21}. \tag{11}
\]
Consider the ODE-LTI (5) with the choice of \( \tilde{A}, G, \tilde{C}, \tilde{D} \) as defined in (10) and (11). We claim that for any \( (x, u) \in L^1_{loc}(I, \mathbb{R}^r) \times L^1_{loc}(I, \mathbb{R}^m), (x, u) \in B_I(E, A, B) \) if and only if, the functions \( (p, q) \in L^1_{loc}(I, \mathbb{R}^r) \times L^1_{loc}(I, \mathbb{R}^{r+m}) \) defined by
\[
\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} T^{-1} & 0 \\ I_m \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{12}
\]
are such that \( p \in AC(I, \mathbb{R}^r), \tilde{p} = \tilde{A}p + Gq \ a.e. \) and \( 0 = \tilde{C}p + \tilde{D}q \) holds for all a.e.. Indeed, notice that \( SEx = SETT^{-1}x = (p^T, 0)^T \). Hence, \( Ex \) is absolutely continuous if and only if \( p \) is absolutely continuous. Furthermore, notice that
\[
S(Ax + Bu) = \begin{bmatrix} \tilde{A}p + Gq \\ \tilde{C}p + \tilde{D}q \end{bmatrix}.
\]
Hence, \( \frac{dEx}{dt} = Ax + Bu \ a.e. \) if and only if
\[
\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = S \frac{dEx}{dt} = S(Ax + Bu) = \begin{bmatrix} \tilde{A}p + Gq \\ \tilde{C}p + \tilde{D}q \end{bmatrix} \ a.e.
\]
Finally note that \( (x, u) \in B_I(E, A, B) \) if and only if \( Ex \) is absolutely continuous and \( \frac{dEx}{dt} = Ax + Bu \ a.e. \). The desired linear system \( \mathcal{S} = (A_I, B_I, C_I, D_I) \) may be obtained as follows. Let \( \tilde{F} \) and \( L \) be the matrices from Theorem
such that

\[ \hat{p} \]

It is easy to see that with this choice, \( S \) is the set of weakly unobservable states of \( (p,g) \).

From Theorem 4 and the discussion above it then follows that for any \((p,g)\) be a minimal realization of \((A,G,\hat{C},\hat{D})\) such that \( p \in AC(I, \mathbb{R}^r) \), \( p(t) \in V, \ t \in I, \ \hat{p} = (A + GF)p + GLg \) a.e., it holds that \( \hat{C}p + \hat{D}(\hat{F}p + Lg) = 0 \) and hence \((x^T, u^T) = \hat{C}p + \hat{D}g \) a.e. Consider a basis \( b_1, \ldots, b_n \) of \( \mathbb{R}^r \) such that \( b_1, \ldots, b_n \) span \( V \). Let \( \mathcal{S} \in \mathbb{R}^{r \times r} \) be the corresponding basis transformation, i.e. \( \mathcal{S}^{-1} = [b_1, \ldots, b_r] \). Let \( D_l = D \) and \( A_l, B_l, C_l \) be the matrix representations of the linear maps \((A+GF)|_V : V \to V, \hat{C}|_V : V \to \mathbb{R}^{n+m}, GL : \mathbb{R}^k \to \mathbb{R}^n \) respectively in the basis \( b_1, \ldots, b_n \). That is,

\[
A_l = \begin{bmatrix} I_n & 0 \end{bmatrix} \mathcal{S}(A + GF)\mathcal{S}^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad C_l = \hat{C}\mathcal{S}^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix},
\]

\[
B_l = \begin{bmatrix} I_n & 0 \end{bmatrix} \mathcal{S}GL.
\]

It is easy to see that with this choice, \( \mathcal{S} = (A_l, B_l, C_l, D_l) \) satisfies Definition 2.

**Remark 1** Notice that the dimension \( \hat{n} \) of the associated linear system constructed in the proof of Theorem 2 satisfies \( \hat{n} \leq \text{Rank} E \leq \max\{c, n\} \).

**Remark 2** (Comparison with [34]) The system \((A_1^*, A_2^*)\) described in [34, Proposition 3] is related to the ODE-LTI constructed in the proof of Theorem 2, see [19] for a detailed explanation.

**Remark 3** Recall from [5] that the augmented Wong sequence is defined as follows \( V_0 = \mathbb{R}^n, V_{i+1} = A^{-1}(EV_i + \text{im} B), \) and that the limit \( V_* = \bigcap_{i=0}^{\infty} V_i \) is achieved in a finite number of steps: \( V_* = V_k \) for some \( k \in \mathbb{N} \). It is not difficult to see that from the proof of Theorem 2 correspond to the limit \( V_* \) of the augmented Wong sequence \( V_i \) for the DAE (4): \( V_* = \{ (p,Fp + Lq)^T | p \in V \} \). Hence, if \((A_l,B_l,C_l,D_l)\) is an ODE-LTI associated with (4), then \( V_* = \text{im} \begin{bmatrix} C_s \\ D_s \end{bmatrix} \), where \( C_s, D_s \) are the matrices formed by the first \( n \) rows of \( C_l \) and \( D_l \) respectively. In [4] a relationship between the quasi-Weierstrass form of regular DAEs and space \( V_* \) for \( B = 0 \) was established. This indicates that there might be a deeper connection between quasi-Weierstrass forms and associated linear systems. The precise relationship remains a topic of future research.

Before presenting the proof of Theorem 3, we present the proof of Corollary 4, as it yields Theorem 3.

**PROOF.** [Proof of Corollary 4] Consider an ODE-LTI system \((A_l,B_l,C_l,D_l)\) associated with (4). From [29, Theorem 4.3] it follows \((A_l,B_l,C_l,D_l)\) is a minimal realization of \( B_r(E,A,B) \), if and only if \( V_* = 0 \), where \( V_* \) is the set of weakly unobservable states of \((A_l,B_l,C_l,D_l)\). Let \( C_s, D_s \) be the matrices formed by the first \( n \) rows of \( C_l \) and respectively \( D_l \). For any \( p_0 \in V_* \), there exist \( (q,p) \in L_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k) \times AC(\mathbb{R}, \mathbb{R}^n) : p(0) = p_0, \hat{p}(t) = A_l p(t) + B_l q(t), C_l p(t) + D_l q(t) = 0 \) a.e.. This implies \( E(C_s p(t) + D_s q(t)) = E C_s p(t) = 0 \) a.e. and by continuity of \( p(t) \) this implies \( E C_s p(t) = 0, t \in I \). In particular, \( p_0 = p(0) = (E C_s)^+(C_s p(0)) = 0 \). That is, \( V_* = 0 \) and \((A_l,B_l,C_l,D_l)\) is minimal.

Let \((A,B,C,D)\) be a minimal realization of \( B_r(E,A,B) \), such that either \( \begin{bmatrix} B \\ D \end{bmatrix} \) is full column rank or it is zero. Let \((A_l,B_l,C_l,D_l)\) be an ODE-LTI system associated with (4). From the discussion above it follows that \((A_l,B_l,C_l,D_l)\) is a minimal realization of \( B_r(E,A,B) \). From the proof of [29, Theorem 7.1] it follows that there exists a nonsingular matrix \( T \) and a matrix \( K \) such that \( A_l = T (A_l + B_l K) T^{-1} \), im \( \begin{bmatrix} TB_l \\ D_l \end{bmatrix} = \text{im} \begin{bmatrix} B \\ D \end{bmatrix} \) and \( (C_l + D_l K) T^{-1} = C \). Hence,
if \( D_l = 0, B_l = 0 \), then \( B = 0, D = 0 \) and both \( \begin{bmatrix} TB_l & D_l \end{bmatrix} \) and \( \begin{bmatrix} B & D \end{bmatrix} \) have one column. Thus, with \( U = 1 \), \( TB_l U = B \), \( D_l U = D \) holds. Otherwise, since both \( \begin{bmatrix} TB_l & D_l \end{bmatrix} \) and \( \begin{bmatrix} B & D \end{bmatrix} \) are full column rank, \( \begin{bmatrix} TB_l & D_l \end{bmatrix} \) and \( \begin{bmatrix} B & D \end{bmatrix} \) have the same number of columns and there exists an invertible matrix \( U \) such that \( U = 1 \), \( TB_l U = B \), \( D_l U = D \). Hence, \((A, B, C, D)\) is feedback equivalent with \((A_l, B_l, C_l, D_l)\). Using feedback equivalence, it is easy to see that \((A, B, C, D)\) satisfies all the conditions of Definition 2.

**PROOF.** [Theorem 3] Theorem 3 is a direct consequence of the proof of Corollary 4.

**PROOF.** [Proof of Corollary 1] If \( z \) is differentiably consistent, then there exists a solution \((x, u)\) of (4) on \( I \) for some interval \( I \), such that \( 0 \in I \) and \( Ex(0) = z \). Then there exist \((v, g) \in AC(I, \mathbb{R}^k) \times L_{loc}^1(I, \mathbb{R}^k) \) such that \( \dot{v} = A_l v + B_l g \) and \((x^T, u^T)^T = C_l v + D_l g\) a.e. In particular, \( Ex(t) = EC_s v(t) \) for almost all \( t \in I \), and hence, by continuity of \( Ex \) and \( v \), \( Ex(t) = EC_s v(t) \) for all \( t \in I \), where \( C_s \) is the matrix formed by the first \( n \) rows of \( C_l \). Therefore, \( z = Ex(0) = EC_s v(0) \) and hence \( z \in \text{im} EC_s \). Conversely, if \( z = EC_s v_0 \), then let \( v \) be the solution of \( \dot{v} = A_l v \), \( v(0) = v_0 \) on \( I \) and set \((x^T, u^T)^T = C_s v \). Then \( Ex(0) = EC_s v(0) = EC_s v_0 = z \) and \((x, u)\) is a solution of (4) on \( I \), i.e. \( z \) is differentiably consistent.

**PROOF.** [Proof of Corollary 2] Consider an ODE-LTI \((A_l, B_l, C_l, D_l)\) which is an associated ODE-LTI for the DAE-LTI (4). If \( z \) is differentiably consistent, then \( z \in \text{im} EC_s \). Let \( p \in AC(I, \mathbb{R}^n) \) be the solution of the differential equation \( \dot{p} = A_p p, p(0) = M(z) \). Then \( p \) is smooth. From the properties of the associated ODE-LTI it then follows that \((x^T, u^T)^T = C_l p \) is a solution of (4) which satisfies \( Ex(0) = z \). Moreover, as \( x \) and \( u \) are linear functions of \( p \), they are also smooth.

**PROOF.** [Proof of Corollary 3] Recall from [5, Section 2] that \( \mathcal{V}^{\text{diff}}_{[E,A,B]} \) is the set of differentiably consistent initial conditions: \( x_0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]} \) if and only if there exists a solution \((x, u)\) of (4) on \( \mathbb{R} \) such that \( Ex(0) = Ex_0 \) and \( x \) is absolutely continuous. From Corollary 2 it follows that \( \mathcal{V}(E, A, B) = \mathcal{V}^{\text{diff}}_{[E,A,B]} \). The statement follows now from [5, Corollary 4.3].

### 3 Application to finite and infinite horizon LQ problem for DAEs

In this section we present the application of the results of Section 2 to LQ control of DAE-LTI. We will start by stating the problem formally. Consider a DAE-LTI of the form (4) We define the set of solutions which satisfy the boundary condition \( Ex(0) = z \).

**Definition 5** Let \( z \in \mathcal{V}(E, A, B) \). For \( t_1 > 0 \), denote by \( \mathcal{D}_z(t_1) \) the set of all solutions \((x, u)\) of (4) on \([0, t_1]\) such that \( Ex(0) = z \). Likewise, define \( \mathcal{D}_z(\infty) \) as the set of all solutions \((x, u)\) of (4) on \([0, +\infty)\) such that \( Ex(0) = z \).

From Corollary 2 it follows that \( \mathcal{D}_z(t_1), t_1 > 0, \mathcal{D}_z(\infty) \) are not empty for any \( z \in \mathcal{V}(E, A, B) \).

Take symmetric \( R \in \mathbb{R}^{m \times m}, Q \in \mathbb{R}^{n \times n}, Q_0 \in \mathbb{R}^{c \times c} \) and assume that \( R > 0, Q > 0, Q_0 \geq 0 \). Fix an initial state \( x_0 \in \mathbb{R}^n \). For any trajectory \((x, u) \in \mathcal{D}_z(t), t \geq t_1 \) define the cost functional

\[
J_{t_1}(x, u) = x(t_1)^T E^T Q_0 E x(t_1) + \int_0^{t_1} (x^T(s) Q x(s) + u^T(s) R u(s)) ds.
\]  

(13)

Note that in (13) \( x, u \) may be defined on an interval larger than \([0, t_1]\), but \( J_{t_1}(x, u) \) depends only on the restriction of \( x \) and \( u \) to \([0, t_1]\). Moreover, \( J_{t_1}(x, u) \) need not be finite, as \( x|_{[0,t_1]} \) and \( u|_{[0,t_1]} \) may not belong to \( L^2([0,t_1], \mathbb{R}^n) \) respectively \( L^2([0,t_1], \mathbb{R}^m) \).
Problem 1 (Finite-horizon optimal control) Consider a differentiably consistent initial state \( z \in \mathcal{V}(E, A, B) \). The problem of finding \((x^*, u^*) \in \mathcal{D}_z(t_1)\) such that:

\[
J_{t_1}(x^*, u^*) = J_{t_1}^* \overset{def}{=} \inf_{(x, u) \in \mathcal{D}_z(t_1)} J_{t_1}(x, u) < +\infty
\]

is called the finite-horizon optimal control problem for the initial state \( z \) and \((x^*, u^*) \) is called the solution of the finite-horizon optimal control problem.

Clearly, the optimal solution \((x^*, u^*)\) should be square integrable (i.e., should belong to \( L^2([0, t_1], \mathbb{R}^n) \times L^2([0, t_1], \mathbb{R}^m) \)) and when calculating \( J_{t_1}^* \), infimum should be taken only over \( L^2([0, t_1], \mathbb{R}^n) \times L^2([0, t_1], \mathbb{R}^m) \cap \mathcal{D}_z(t_1) \), since for all other solutions the cost function is infinite.

Problem 2 (Infinite horizon optimal control) Consider a differentiably consistent initial state \( z \in \mathcal{V}(E, A, B) \). For every \((x, u) \in \mathcal{D}_z(\infty)\), define

\[
J_\infty(x, u) = \limsup_{t \to \infty} J_t(x, u).
\]

The infinite horizon optimal control problem for the initial state \( z \) is the problem of finding \((x^*, u^*)\) such that \((x^*, u^*) \in \mathcal{D}_z(\infty)\) and

\[
J_\infty(x^*, u^*) = J_{\infty}^* < +\infty
\]

\[
J_{\infty}^* \overset{def}{=} \limsup_{t_1 \to \infty} \inf_{(x, u) \in \mathcal{D}_z(t_1)} J_{t_1}(x, u).
\]

The pair \((x^*, u^*)\) will be called the solution of the infinite horizon (optimal) control problem for the initial state \( z \).

Note that the optimal solution \((x^*, u^*)\) of the infinite horizon problem should belong to \( L^2([0, +\infty), \mathbb{R}^n) \times L^2([0, +\infty), \mathbb{R}^m) \). Note that \( L^2([0, +\infty), \mathbb{R}^p) \) is a subset of \( L^2_{loc}([0, +\infty), \mathbb{R}^p) \), \( p = n, m \) and hence \((x^*, u^*) \in L^2([0, +\infty), \mathbb{R}^n) \times L^2([0, +\infty), \mathbb{R}^m) \) does not conflict with the definition of \( \mathcal{D}_z(\infty) \).

Remark 4 The proposed formulation of the infinite horizon control problem is not the most natural one. It also makes sense to look for solutions \((\tilde{x}, \tilde{u}) \in \mathcal{D}_z(\infty)\) which satisfy \( J_\infty(\tilde{x}, \tilde{u}) = \inf_{(x, u) \in \mathcal{D}_z(\infty)} J_\infty(x, u) \). The latter means that the cost induced by \((\tilde{x}, \tilde{u})\) is the smallest among all the trajectories \((x, u)\) which are defined on the whole time axis. It is easy to see that if \((x^*, u^*)\) is a solution of Problem 2, then \( J_\infty(x^*, u^*) = J_{\infty}^* = \inf_{(x, u) \in \mathcal{D}_z(\infty)} J_\infty(x, u) \), i.e., the solution of Problem 2 yields the minimal cost among all the solutions \((x, u)\) of the DAE-LTI which satisfy \( E\tilde{x} = z \) and which are defined on the whole time axis. Another option is to use \( \lim \) instead of \( \limsup \) in the definition of \( J_\infty(x^*, u^*) \) and in (14). In fact, the solution we are going to present remains valid if we replace \( \limsup \) by \( \lim \).

Remark 5 (Derivatives of inputs in \( J \)) Note that the cost function \( J_{t_1}(x, u) \) does not contain explicitly the derivatives of \( x \) and \( u \). However, as it is well known from solution theory of DAEs, derivatives of \( u \) can implicitly appear in the state \( x \) and hence in the cost function. It is especially obvious if one computes the Kronecker canonical form of the DAE at hand and rewrites the cost function in the new coordinates. We stress that in our framework the state and input of the DAE-LTI are linear functions of the output of the associated ODE-LTI. Thus, if the DAE-LTI’s state depends on derivatives of the input this will be taken into account implicitly. As a result, the cost \( J \) will also include this relation implicitly (it is made clear in (26) where \( J \) is reformulated in terms of the associated DAE-LTI).

The rest of the section is organized as follows. In §3.1 we present the main results and in §3.2 we present their proofs.
3.1 Main results

We start by presenting a solution to the finite horizon case. To this end, let \( \mathcal{S} = (A_l, B_l, C_l, D_l) \) be an ODE-LTI associated with the DAE-LTI (4). Consider the following differential Riccati equation

\[
\begin{align*}
\dot{P}(t) &= A_l^T P(t) + P(t) A_l - K^T(t)(D_l^T SD_l)K(t) + C_l^T SC_l \\
\dot{P}(0) &= (EC_s)^T Q_0 EC_s, \quad S = \text{diag}(Q, R) \\
K(t) &= \begin{cases} 
(D_l^T SD_l)^{-1} (B_l^T P(t) + D_l^T SC_l) & \text{if } D_l \neq 0 \\
0 & \text{if } D_l = 0
\end{cases}
\end{align*}
\]

where \( C_s \) is the matrix formed by the first \( n \) rows of \( C_l \). Note that either \( D_l \) is full column rank, and hence by positive definiteness of \( Q, R, D_l^T SD_l \) is invertible, or \( B_l = 0, D_l = 0, k = 1 \). Hence, (15) is always well-defined. For any \( z \in \mathcal{V}(E, A, B) \), let \((x^*, u^*)\) be defined as

\[
(x^T(s), u^T(s))^T = (C_l - D_lK(t - s))v(s), \quad \dot{v}(s) = (A_l - B_lK(t - s))v(s) \quad \text{and} \quad v(0) = \mathcal{M}(z).
\]

Furthermore, define \( K_f(s) = (C_u + D_uK(t - s))ME \), where \( C_u \) and \( D_u \) are the matrices formed by the last \( m \) rows of \( C_l \) and \( D_l \) respectively. Define the matrices:

\[
K_1(t) = (C_l - D_lK(t_1 - t))ME - \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
K_2(t) = \begin{bmatrix} 0 & 0 \\ 0 & -I_m \end{bmatrix}.
\]

**Theorem 5** With the notation above, \((x^*, u^*)\) is a solution of the finite horizon optimal control problem for the interval \([0, t]\) and the initial state \( z \). The optimal value of the cost function is

\[
J_f^* = J_l(x^*, u^*) = (\mathcal{M}(z))^T P(t)(\mathcal{M}(z)).
\]

Furthermore, \( u^*(s) = K_f(s)x^*(s), s \in [0, t] \), and \((x^*, u^*) = (x, u)\) is the unique (up to modification on a set of measure zero) solution of

\[
\begin{align*}
\frac{d(EX(t))}{dt} &= Ax(t) + Bu(t), \\
K_1(t)x(t) + K_2(t)u(t) &= 0,
\end{align*}
\]

on \([0, t_1]\), such that \( EX(0) = z \).

Next, we present the solution to the infinite horizon control problem. Just like in the classical case, we will need a certain notion of stabilizability for solvability of the infinite horizon LQ control problem.

**Definition 6 (Behavioral stabilizability)** The DAE-LTI (4) is said to be behaviorally stabilizable from \( z \in \mathcal{V}(E, A, B) \), if there exists \((x, u) \in \mathcal{D}_{\infty}\) such that \( \lim_{t \to \infty} x(t) = 0 \).

Behavior stabilizability from \( z \) can be interpreted in terms of the associated ODE-LTI as follows. Let \( \mathcal{S} = (A_l, B_l, C_l, D_l) \) be an ODE-LTI associated with (4) and let \( \mathcal{M} \) be the corresponding state map. Let \( \mathcal{V}_g \) denote the stabilizability subspace of \( \mathcal{S} \). Recall from [27] that \( \mathcal{V}_g \) is the set of all initial states \( p_0 \) of \( \mathcal{S} \), for which there exists an input \( g \) such that the corresponding state trajectory \( p \) starting from \( p_0 \) has the property that \( \lim_{t \to \infty} p(t) = 0 \).

\[1\] I.e., for any \((x, u) \in AC([0, t], \mathbb{R}^n) \times L^1_{\text{loc}}([0, t], \mathbb{R}^m)\) such that \( EX(0) = z \), \( \frac{d(EX(t))}{dt} = Ax(t) + Bu(t) \) a.e, and \( K_1(t)x(t) + K_2(t)u(t) = 0 \) a.e., if and only if \((x, u) = (x^*, u^*)\) a.e.
Lemma 1 The DAE-LTI (4) is behaviorally stabilizable from \( z \in \mathcal{V}(E,A,B) \) if and only if \( \mathcal{M}(z) \) belongs to the stabilizability subspace \( \mathcal{V}_g \) of \( \mathcal{S} \).

In order to solve the infinite horizon control problem for DAE-LTIs, we reformulate it as an infinite horizon control problem for the associated ODE-LTIs. However, for ODE-LTIs, infinite horizon LQ control problems can be solved only for stabilizable ODE-LTIs. For this reason, we will need to define the restriction of an associated ODE-LTI to its stabilizability subspace. More precisely, consider the ODE-LTI \( \mathcal{S} = (A_l, B_l, C_l, D_l) \) associated with (4) and consider its stabilizability subspace \( \mathcal{V}_g \). From [8] it then follows that \( \mathcal{V}_g \) is \( A_l \)-invariant and \( \text{im} B_l \subseteq \mathcal{V}_g \). Hence, there exists a basis transformation \( T \) such that \( T(\mathcal{V}_g) = \text{im} \begin{bmatrix} I_l & 0 \\ 0 & 0 \end{bmatrix} \), \( l = \dim \mathcal{V}_g \) and in this new basis,

\[
TA_l T^{-1} = \begin{bmatrix} A_g & * \\ 0 & * \end{bmatrix}, \quad TB_l = \begin{bmatrix} B_g \\ 0 \end{bmatrix}, \quad C_l T^{-1} = \begin{bmatrix} C_g^T \\ * \end{bmatrix}^T,
\]

\( A_g \in \mathbb{R}^{l \times l}, B_g \in \mathbb{R}^{l \times k}, C_g \in \mathbb{R}^{(n+m) \times l} \). Denote by \( \mathcal{S}_g = (A_g, B_g, C_g, D_g) \), where \( D_g = D_l \).

Definition 7 (Stabilizable associated ODE-LTI) We call \( \mathcal{S}_g \) a stabilizable ODE-LTI associated with (4) and we call \( \mathcal{M}_g = \begin{bmatrix} I_l & 0 \end{bmatrix} T \mathcal{M} \) the associated state map.

The ODE-LTI \( \mathcal{S}_g \) represents the restriction of \( \mathcal{S} \) to the subspace \( \mathcal{V}_g \). It follows that \( \mathcal{S}_g \) is stabilizable. Moreover, since all associated ODE-LTIs of (4) are feedback equivalent, then all associated stabilizable ODE-LTIs of (4) are also feedback equivalent. Consider a stabilizable ODE-LTI \( \mathcal{S}_g = (A_g, B_g, C_g, D_g) \) associated with (4), and the corresponding state map \( \mathcal{M}_g \) and assume that \( \mathcal{M}(z) \in \mathcal{V}_g \). Consider the following algebraic Riccati equation:

\[
0 = PA_g + A_g^T P - K^T (D_g^T SD_g) K + C_g^T SC_g, \\
K = \begin{cases} 
(D_g^T SD_g)^{-1} (B_g^T P + D_g^T SC_g) & \text{if } D_l \neq 0 \\
0 & \text{if } D_l = 0.
\end{cases}
\tag{20}
\]

\[
S = \text{diag}(Q,R)
\]

Note that either \( D_l \) is full column rank, and hence by positive definiteness of \( Q,R, D_l^T S D_l \) invertible, or \( B_l = 0, D_l = 0, k = 1 \). In the former case, \( D_g = D_l \) is full column rank, in the latter case, \( D_g = 0 \) and \( B_g = 0 \) Hence, (15) is always well-defined.

Lemma 2 The algebraic Riccati equation (20) has a unique symmetric solution \( P \) > 0 and \( A_g - B_g K \) is a stable matrix.

Consider now the tuple \( (x^*, u^*) \in L^2_{loc}([0, +\infty), \mathbb{R}^n) \times L^2_{loc}([0, +\infty), \mathbb{R}^m) \) such that

\[
\dot{v}^* = (A_g - B_g K)v^*, \quad v^*(0) = \mathcal{M}_g(Ex_0), \\
(x^T, u^T)T = (C_g - D_g K)u^*.
\tag{21}
\]

Furthermore, define the following matrices:

\[
K_f = (C_u - D_u K) \mathcal{M}, \\
K_1 = (C_g - D_g K) ME - \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \\
K_2 = \begin{bmatrix} 0 & 0 \\ 0 & -I_m \end{bmatrix},
\tag{22}
\]

where \( C_u \) and \( D_u \) are the matrices formed by the last \( m \) rows of \( C_g \) and \( D_g \) respectively. We can now state the following.
Theorem 6 The following are equivalent:

• (i) The infinite horizon optimal control problem is solvable for $z \in \mathcal{V}(E, A, B)$
• (ii) The DAE-LTI (4) is behaviorally stabilizable from $z$
• (iii) $J^* = \limsup_{t \to \infty} \inf_{(x,u) \in \mathcal{G}_z(t)} J(t)$

If either of the conditions (i) – (ii) hold, then $(x^*, u^*)$ from (21) is a solution of the infinite horizon optimal control problem for the initial state $z$. Moreover,

$$J^*_\infty = J_\infty(x^*, u^*) = (M_z Ez)^T P M_z Ez,$$

and

$$u^* = K_f x^*, (x^*, u^*) \text{ is a solution of the DAE-LTI}$$

$$\frac{d(Ex(t))}{dt} = Ax(t) + Bu(t),$$

on $[0, +\infty)$ such that $Ex^*(0) = z$, and if $(x, u)$ is a solution of (24) on $[0, +\infty)$ such that $Ex(0) = z$, then $(x, u) = (x^*, u^*)$ a.e.

The proof of Theorem 6 implies that in the formulation of optimal control problem, we can replace $\limsup$ by $\lim$.

Note that the existence of solution for Problem 1 and Problem 2 and its computation depend only on the matrices $(E, A, B, Q, R, Q_0)$. Indeed, an ODE-LTI $\mathcal{J}$ associated with $(E, A, B)$ can be computed from $(E, A, B)$, and the solution of the associated LQ problem can be computed using $\mathcal{J}$ and the matrices $Q, Q_0, R$. Notice that the only condition for the existence of a solution is behavioral stabilizability from $z$, and this can be checked by verifying if $M(z)$ belongs to the stabilizability subspace of $\mathcal{J}$. The latter can be done by an algorithm. The Matlab code for solving Problem 1 and Problem 2 and checking behavioral stabilizability is available at http://sites.google.com/site/mihalypetreczky/.

We would like to conclude this section with a short discussion on the notion of stabilizability we proposed. First, there are several equivalent ways to define behavioral stabilizability. Below we state some of them.

Corollary 5 For any $z \in \mathcal{V}(E, A, B)$ the following are equivalent.

• (i) (4) is behaviorally stabilizable from $z$
• (ii) there exist $(x, u) \in \mathcal{G}_z(\infty)$ such that $\lim_{t \to \infty} Ex(t) = 0$
• (iii) for any $(x, u) \in \mathcal{B}_E(E, A, B)$ such that $Ex(0) = z$ and $x$ is absolutely continuous, there exist $(x_o, u_o) \in \mathcal{B}_E(E, A, B)$ such that $(x(t), u(t)) = (x_o(t), u_o(t))$ for all $t < 0$, $Ex(0) = z$ and $\lim_{t \to \infty} x_o(t) = 0$, $\lim_{t \to \infty} u_o(t) = 0$ and $x_o$ is absolutely continuous.

In fact, Part (iii) of Corollary 5 implies that if (4) is behaviorally stabilizable for all $z \in \mathcal{V}(E, A, B)$, then (4) is behaviorally stabilizable in the sense of [3, 5]. Hence,

Corollary 6 ([5, Corollary 4.3], [3, Proposition 3.3]) The DAE-LTI (4) is stabilizable for all $z \in \mathcal{V}(E, A, B)$ if and only if

$$\forall \lambda \in \mathbb{C}, \text{Re} \lambda \geq 0 : \text{rank} \left[ sE - A, B \right] = \text{nrank} \left[ sE - A, B \right].$$

Here, $\text{nrank} \left[ sE - A, B \right]$ denotes the rank of the polynomial matrix $\left[ sE - A, B \right]$ over the quotient field of polynomials in the variable $s$.

Note that behavior stabilizability from all $z \in \mathcal{V}(E, A, B)$ is equivalent to behavioral stabilizability in the sense of [5], and the latter is equivalent to existence of an algebraic constraint which stabilizes the closed-loop system, see [5]. By Theorem 6, behavior stabilizability from all $z \in \mathcal{V}(E, A, B)$ is equivalent to the existence of a solution of Problem 2 for all $z \in \mathcal{V}(E, A, B)$. The resulting optimal state trajectory $x^*$ converges to zero, and it can be enforced by adding the algebraic constraint $K_1 x + K_2 u = 0$ to the original DAE-LTI. In fact, $K_1 x + K_2 u = 0$ from Theorem 6 is a particular instance of a stabilizing algebraic constraint from [3, 5]. However, as it was already pointed out in [3, 5], behavioral stabilizability does not imply existence of a stabilizing feedback. Below we present an example which is behaviorally stabilizable but cannot be stabilized by a state feedback.
Example 1 Consider the following DAE-LTI

\[
\begin{align*}
\dot{x}_1 &= x_1 + u, \\
\dot{x}_2 &= x_2 + x_3.
\end{align*}
\]

Since the second equation does not depend on u, no matter how we choose the feedback \( g \) for some function \( g \), it will not influence \( x_2 \). That is, there is no chance to enforce any restriction on \( x_2 \) by using the control input only. However, optimal and stabilizing control is still possible. Consider the matrices \( Q = I_3, R = 1, Q_0 = I_2 \) and the corresponding optimal control problem (Problem 2). An associated ODE-LTI \((A_l, B_l, C_l, D_l)\) can be chosen so that

\[
A_l = I_2, B_l = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C_l = \begin{bmatrix} I_2 \\ 0 \end{bmatrix}, D_l = \begin{bmatrix} 0 \\ I_2 \end{bmatrix}. \]

This system is clearly controllable and hence stabilizable, and it can be taken as a stabilizable ODE-LTI associated with the DAE-LTI. The solution of the Riccati equation and the corresponding matrices \( K_1 \) and \( K_2 \) can readily be computed.

The optimal control problem described in Example 1 arises when trying to solve the problem of estimating the state \( z_1 \) of the following noisy DAE with the output \( y: \dot{z}_1 = -z_1 + f_1, \dot{z}_2 = -z_2 + f_2, z_2 = f_3, y = z_1 + \eta \). Here \( f_1, f_2, f_3, \eta \in L^2([0, +\infty), \mathbb{R}) \) are deterministic noise signals such that: \( z_1(0) + \int_{0}^{\infty} f_1^2(s) + f_2^2(s) + f_3^2(s) + \eta^2(s)ds \leq 1 \), i.e. the energy of \( f_1, f_2, f_3, \eta \) is bounded and the unknown initial state \( z_1(0) \) is bounded. Then according to [37,38], in order to construct an estimate \( \hat{z}_1 \) of \( z_1 \) from \( y \) with the minimal worst-case estimation error, one needs to solve the LQ problem of Example 1. Conversely, if \( \hat{z}_1 \) exists, then the LQ control problem from Example 1 will have a solution. That is, even for such toy models (if \( \eta = 0 \), the problem is trivial), the state estimation problem yields an optimal control problem which cannot be solved by state feedback alone.

Example 1 shows that DAE-LTIs which are behaviorally stabilizable but do not admit a stabilizing feedback occur naturally. It shows that the lack of a stabilizing feedback control is not a shortcoming of the definition, but a sign that concept of the feedback control might be too restrictive for DAE-LTIs. In our opinion, one should consider more general controllers, for example, controllers which are represented by algebraic constraints \( K_1x + K_2u = 0 \). The latter can be viewed as a controller, if we follow the philosophy of J.C. Willems [30, 31]. Note that many physical control devices cannot be described as feedback controllers, [30, 31], including such simple example as mass-spring-dumper systems and electrical circuits. Note that a classical feedback \( u = Kx \) is just a specific case of a controller enforcing algebraic constraints: it can be represented as \( u - Kx = 0 \).

The fact that we consider DAE-LTIs which cannot be optimized or even stabilized by state feedback explains why our results differ from [13–16]. In [13–16] sufficient conditions for existence of an optimal state feedback control were presented. In particular, the conditions of [13–16] imply existence of a stabilizing state feedback control law. These conditions cannot be satisfied by systems which cannot be stabilized by state feedback alone. The system from Example 1 is one such system, and for that system the conditions of [13–16] never hold, no matter which quadratic cost function we choose.

3.2 Proofs

In order to present the proofs of Theorem 5 and Theorem 6, we rewrite Problems 1 – 2 as LQ control problems for ODE-LTIs. To this end, consider an ODE-LTI \( \mathcal{J} = (A_l, B_l, C_l, D_l) \) and let \( \mathcal{M} = (EC_s)^+ \) be the corresponding state map. Recall that \( C_s \) is the first \( n \) rows of \( C_l \). Consider the following linear quadratic control problem. For every initial state \( v_0 \), for every interval \( I \) containing \([0, t_1]\) and for every \( g \in \mathcal{H}^1_{loc}(I, \mathbb{R}^k) \) define the cost functional \( \mathcal{J}_I(v_0, g) \):

\[
\begin{align*}
\mathcal{J}_I(v_0, g) &= v^T(t)C^TE^TQ_0ECsv(t) + \\
&+ \int_{0}^{t} v^T(s) \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} v(s)ds, \\
\dot{v} &= Av + Bfg, \quad v(0) = v_0, \\
\nu &= Cv + Dfg.
\end{align*}
\]
For any \( g \in L^1_{loc}([0, \infty), \mathbb{R}^k) \) and \( v_0 \in \mathbb{R}^n \), define
\[
\mathcal{J}_\infty(v_0, g) = \limsup_{t \to \infty} \mathcal{J}_t(v_0, g).
\] (27)

In our next theorem we prove that the problem of minimizing the cost function \( J_t \) for (4) is equivalent to minimizing the cost function \( \mathcal{J}_t \) for the associated ODE-LTI \( \mathcal{S} \).

**Theorem 7** With the notation above, let \( z \in \mathcal{V}(E, A, B) \), \( I = [0, t] \), \( t > 0 \) or \( I = [0, \infty) \). For \( g \in L^1_{loc}(I, \mathbb{R}^k) \) denote by \( \nu_{\mathcal{S}}(v_0, g) \) the output trajectory of the associated ODE-LTI \( \mathcal{S} \), which corresponds to the initial state \( v_0 \) and input \( g \).

(i) For any \( g \in L^1_{loc}(I, \mathbb{R}^k) \), and for any \((x, u) \in \mathcal{D}_x(t)\), such that \((x^T, u^T)^T = \nu_{\mathcal{S}}(M(z), g)\),
\[
J_t(x, u) = \mathcal{J}_t(M(z), g),
\]
\[
J_\infty(x, u) = \mathcal{J}_\infty(M(z), g).
\] (28)

(ii) For all \( t \in (0, \infty) \), \((x^*, u^*) \in \mathcal{D}_x(t)\) is a solution of the finite horizon optimal control problem if and only if there exists \( g^* \in L^1([0, t], \mathbb{R}^k) \) such that \((x^{*T}, u^{*T})^T = \nu_{\mathcal{S}}(M(z), g^*)\),
\[
\mathcal{J}_t(v_0, g^*) = \inf_{g \in L^1_{loc}([0, t], \mathbb{R}^k)} \mathcal{J}_t(v_0, g) < +\infty.
\] (29)

(iii) The tuple \((x^*, u^*) \in \mathcal{D}_x(\infty)\) is a solution of the infinite horizon optimal control problem if and only if there exists an input \( g^* \in L^1_{loc}([0, \infty), \mathbb{R}^k) \) such that \((x^{*T}, u^{*T})^T = \nu_{\mathcal{S}}(M(z), g^*)\), and
\[
\mathcal{J}_\infty(v_0, g^*) = \limsup_{t \to \infty} \inf_{g \in L^1_{loc}([0, t], \mathbb{R}^k)} \mathcal{J}_t(v_0, g) < +\infty.
\] (30)

**PROOF.** [Proof of Theorem 7] Equation (28) follows by routine manipulations and by noticing that the first \( n \) rows of \( C_s v(t) + D_s g(t) \) equal \( C_s v(t) + D_s g(t) \) and as \( ED_s = 0 \) (see Definition 2 for the definitions of \( C_s \) and \( D_s \)), \( E(C_s v(t) + D_s g(t)) = EC_s v(t) \). The rest of theorem follows by noticing that for any element \((x, u) \in \mathcal{D}_x(t)\) or \( \mathcal{D}_x(\infty) \), there exist \( g \in L^1_{loc}(I, \mathbb{R}^k) \) such that for \( v = ME x, (x^T, u^T)^T = Cv + Dg \), a.e. \( \dot{v} = Au + B v, v(0) = ME x(0) = M z, \) and conversely, for any \((v, g) \in AC(I, \mathbb{R}^n) \times L^1_{loc}(I, \mathbb{R}^k) \) such that \( \dot{v} = Au + B g, v(0) = M z, Cv + Dg \in \mathcal{D}_x(t) \) (if \( I = [0, t] \)) or \( Cv + Dg \in \mathcal{D}_x(\infty) \) (if \( I = [0, \infty) \)).

The proof of Theorem 5 can then be derived from the classical results (see [17]).

**PROOF.** [Proof of Theorem 5] By definition of an associated ODE-LTI, there are two possible cases: either \( D_l \) is full column rank or \( D_l = 0 \). This two cases cover all the possibilities. We start with the case when \( D_l \neq 0 \) is full column rank.

Let us first apply the feedback transformation \( g = \hat{F} v + U w \) to \( \mathcal{S} = (A_l, B_l, C_l, D_l) \) with \( U = (D_l^T S D_l)^{-1/2} \) and \( \hat{F} = -(D_l^T S D_l)^{-1} D_l^T S C_l \), as described in [27, Section 10.5, eq. (10.32)]. Note that \( D_l \) is injective and hence \( U \) is well defined. Consider the linear system
\[
\dot{v} = (A_l + B_l \hat{F}) v + B_l U w \quad \text{and} \quad v(0) = v_0.
\] (31)

For any \( w \in L^1_{loc}([0, t], \mathbb{R}^k) \), the state trajectory \( v \) of (31) equals the state trajectory of \( \mathcal{S} = (A_l, B_l, C_l, D_l) \) for the input \( g = \hat{F} v + U w \) and initial state \( v_0 \). Moreover, from Theorem 4 it follows that all inputs \( g \in L^1_{loc}([0, t], \mathbb{R}^k) \) of \( (A_l, B_l, C_l, D_l) \) can be represented in such a way. Define
\[
\hat{\mathcal{J}}_t(v_0, w) = v^T(t)C_s^T E^T Q_0 E C_s v(t) + \int_0^t (v^T(s)(C_l + D_l \hat{F})^T S(C_l + D_l \hat{F})v(s) + w^T(s) w(s)) ds,
\]
where \( v \) is a solution of (31), and \( C_s \) is the matrix formed by the first \( n \) rows of \( C_t \). It is easy to see that \( \mathcal{F}_t(v_0, g) = \mathcal{F}_t(v_0, w) \) for \( g = Fv + Wu \) and any initial state \( v_0 \) of \( \mathcal{S} \).

Consider now the problem of minimizing \( \mathcal{F}_t(v_0, w) \). The solution of this problem can be found using [17, Theorem 3.7]. Notice that (15) is equivalent to the Ricatti differential equation described in [17, Theorem 3.7] for the problem of minimizing \( \mathcal{F}_t(v_0, w) \). Hence, by [17, Theorem 3.7], (15) has a unique positive solution \( P \), and for the optimal input \( u^* \), \( g^* = Fv^* + Wu^* = -K(t_1 - t)v(t) \) satisfies (29), and \( \dot{v}(t) = (A_t - B_tK(t_1 - t))v(t) \) and \( v(0) = v_0 \). From Theorem 7 and Definition 2 it then follows that \( (x^*T, u^*T)^T = C_tv^* + Dtg^* \) is the solution of the Problem 1 and that (18) holds.

Assume now that \( D_t = 0 \). From the definition of an associated ODE-LTI it then follows that \( B_t = 0 \). Hence, the associated ODE-LTI is in fact an autonomous system. Define now \( \nu \) the partition of unity, \( \nu = \nu \). Let \( \phi \) exists a feedback control law \( D \) noticing that in the proof of the "only if" part of Lemma 1 it is sufficient to assume that \( (4) \) can be shown as follows. Let \( \mathcal{S} \) be the corresponding state map \( \mathcal{S} \) of \( \mathcal{S} \) and \( \mathcal{R} \) of \( \mathcal{S} \) on \( [0, t] \) and initial state \( z \). Finally, notice that the state trajectory \( v(16) \) satisfies \( v(s) = A_t^{s} \mathcal{M}(z) \) and \( (x^*T, u^*T)^T(s) = C_tv^* + Dtg^* \) and hence (18) holds.

Finally, in both cases \( D_t = 0 \) and \( D_t = 0 \), \( u^*(s) = Kf(s)x^*(s) \) for all \( s \in [0, t] \) and all the solutions \( (x, u) \) of (19) on \( [0, t] \) are those which satisfy \( (x^T, u^T)^T(s) = (C_t - D_tF(s))v(s) \) a.e., \( v(s) = (A_t - B_tF(s))v(s) \), \( v(s) = \mathcal{M}(Ex(s)) \), \( s \in [0, t] \). Hence, \( (x^*, u^*) \) is indeed the only solution of (19) such that \( Ex^*(0) = z \).

**PROOF.** [Proof of Lemma 1] "if part" Assume that \( \mathcal{S} = (A_t, B_t, C_t, D_t) \) is stabilizable from \( \mathcal{M}(z) \). Let \( \mathcal{V}_g \) be the stabilizability subspace of \( \mathcal{S} \). It then follows from [27] that \( \mathcal{M}(z) \in \mathcal{V}_g \) and there exists a feedback \( F \) such that the restriction of \( A_t + B_tF \) to \( \mathcal{V}_g \) is stable and hence for any \( v_0 = (A_t + B_tF)v, v(0) = v_0 \), and \( lim_{t \to \infty} v(t) = 0 \). Consider the output \( (x^T, u^T)^T = (C_t - D_tF)v \). It then follows that \( Ex^*(0) = z \) and \( (x, u) \) is a solution of the DAE-LTI. Moreover, \( lim_{t \to \infty} x(t) = lim_{t \to \infty}(C_s + D_sF)v(t) = 0 \). That is, DAE-LTI is stabilizable from \( z \).

"only if part" Assume that the DAE-LTI is stabilizable from \( z \), and let \( (x, u) \in \mathcal{S}(\infty) \) be such that \( lim_{t \to \infty} x(t) = 0 \). It then follows that there exist an input \( g \in L_{loc}^1([0, +\infty), \mathbb{R}^k) \) such that \( \dot{v} = Av + Bg \). \( (x^T, u^T)^T = C_tv + Dtg \) a.e., \( v(0) = \mathcal{M}(Ex_0) \) and \( \mathcal{M}(Ex(t)) = v(t), t \in [0, +\infty). \) In particular, \( lim_{t \to \infty} v(t) = \mathcal{M}(z) \). That is, there exists an input \( g \) such that the corresponding state trajectory \( v \) of \( \mathcal{S} \) starting from \( \mathcal{M}(z) \) converges to zero. But this is precisely the definition of stabilizability of \( (A_t, B_t, C_t, D_t) \) from \( \mathcal{M}(z) \).

**PROOF.** [Proof of Corollary 5] The implications (i) \( \Rightarrow \) (ii) is trivial. The implication (ii) \( \Rightarrow \) (i) follows by noticing that in the proof of the "only if" part of Lemma 1 it is sufficient to assume that \( (x, u) \in \mathcal{S}(\infty) \) is such that \( lim_{t \to \infty} Ex(t) = 0 \).
continuous. Moreover, \((x_o^T(t), u_o^2(t))^T = (C_l + D_f F_l)p_o(t)\) for all \(t \geq 1\) and hence \(x_o(t), u_o(t)\) converge to zero as \(t \to +\infty\).

The implication (iii) \(\implies\) (i) can be shown as follows. Since \(z\) is differentiably consistent, from Corollary 3 it follows that there exist \((x, u) \in B_g(E, A, B)\) such that \(Ex(0) = z\) and \(x\) is differentiable. Then from (iii) it follows that there exist \((x_o, u_o) \in B_g(E, A, B)\) such that \((x_o(t), u_o(t)) = (x(t), u(t))\) for all \(t < 0\), \(Ex(0) = z\) and \(\lim_{t \to -\infty}(x_o^T(t), u_o^2(t))^T = 0\). In particular, \((x_o_{[0, +\infty)}, u_o_{[0, +\infty)}) \in \mathcal{Z}_z(\infty)\) and \(\lim_{t \to +\infty} x_o(t) = 0\), i.e. (4) is behaviorally stabilizable from \(z\). □

**PROOF.** [Proof of Lemma 2] If \(D_g = 0\), then \(B_g = 0\) and as \((A_g, B_g)\) is stabilizable, \(A_g\) is stable. Then existence of \(P = P^T > 0\) satisfying (20) follows from the existence of the observability gramian for a stable linear systems.

Assume now that \(D_g\) is full column rank. Let us apply the feedback transformation \(g = \hat{F}v + Uw\) to \(\mathcal{S}_g = (A_g, B_g, C_g, D_g)\) with \(U = (D_g^TSD_g)^{-1/2}\) and \(\hat{F} = -(D_g^TSD_g)^{-1}D_g^TSC_g\), as described in [27, Section 10.5, eq. (10.32)]. To this end, notice that \((A_g + B_g \hat{F}, B_g U)\) is stabilizable and \((S^{1/2}(C_g + D_g \hat{F}), A_g + B_g \hat{F})\) is observable. Indeed, it is easy to see that stabilizability of \((A_g, B_g)\) implies that of \((A_g + B_g \hat{F}, B_g U)\). Observability of \((S^{1/2}(C_g + D_g \hat{F}), A_g + B_g \hat{F})\) can be derived as follows. Recall from Definition 2 that \(EC_s\) is of full column rank and \(ED_s = 0\). Note that \(D_g = D_t\). Let \(\hat{C_s}\) be the matrix formed by the first \(n\) rows of \(C_g\). Then \(EC_s\) is the the restriction of the map \(E(C_s + D_s \hat{F}) = EC_s\) to \(V_g\), hence \(EC_s\) is of full column rank, if \(EC_s\) is injective. The latter is the case according to Definition 2. Hence, \((C_s + D_s \hat{F}) = \hat{C_s}\) is of full column rank, and thus the pair \((C_s + D_s \hat{F}, A_g + B_g \hat{F})\) is observable.

Consider the ODE-LTI

\[
\dot{v} = (A_g + B_g \hat{F})v + B_g Uw \quad \text{and} \quad v(0) = \hat{v}_0.
\]

(32)

For any \(w \in L^1_{loc}(I, \mathbb{R}^k)\), where \(I = [0, t], 0 < t \in \mathbb{R}\) or \(I = [0, +\infty)\), the state trajectory \(v\) of (32) equals the state trajectory of \(\mathcal{S}_g = (A_g, B_g, C_g, D_g)\) for the input \(g = \hat{F}v + Uw\) and initial state \(\hat{v}_0\). Moreover, all inputs \(g \in L^1_{loc}(I, \mathbb{R}^k)\) of \((A_g, B_g, C_g, D_g)\) can be represented in such a way. Define now

\[
\mathcal{H}_t(\hat{v}_0, w) = v^T(t)(EC_s)^TQ_0EC_s v(t) + \int_0^t \left[w(s)^T w(s) + (v^T(s)(C_g + D_g \hat{F})^T S(C_g + D_g \hat{F}) v(s))ds,
\]

where \(v\) is a solution of (32). Consider now the problem of minimizing \(\lim_{t \to -\infty} \mathcal{H}_t(\hat{v}_0, w)\). Notice that (20) is equivalent to the algebraic Riccati equation described in [17, Theorem 3.7] for the ODE-LTI (32) and for the infinite horizon cost function \(\lim_{t \to -\infty} \mathcal{H}_t(\hat{v}_0, w)\). Hence, by [17, Theorem 3.7], (20) has a unique positive definite solution \(P\), and \(A_g + B_g F - B_g U^T B_g^T P = A_g - B_g K\) is a stable matrix. □

**PROOF.** [Proof of Theorem 6] (i) \(\implies\) (ii) If \((x^*, u^*)\) is a solution of the infinite horizon optimal control problem, then by Theorem 7, there exists an input \(g^* \in L^1_{loc}(0, +\infty), \mathbb{R}^k)\) such that \(\mathcal{J}_\infty(\mathcal{M}(z), g^*) < +\infty\). Let \(v_0 = \mathcal{M}(z)\). We claim that if \(\mathcal{J}_\infty(v_0, g^*) < +\infty\), then \(\lim_{t \to -\infty} v^*(t) = 0\) for the state trajectory \(v^*(t)\) of \(\mathcal{S}\) which corresponds to the input \(g^*\) and starts from \(v_0\). The latter is equivalent to \(v^*(0) = v_0 = \mathcal{M}(Ex) \in V_g\). Let us prove that \(\lim_{t \to -\infty} v^*(t) = 0\). To this end, notice that

\[
\mathcal{J}_\infty(v_0, g^*) < +\infty \implies \int_0^\infty \nu(t)dt < +\infty, \text{ and as } \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \text{ is positive definite, it follows that } \mu \int_0^\infty \nu^T(t)\nu(t)dt < \int_0^\infty \nu(t)dt < +\infty \text{ for some } \mu > 0 \text{ and so } \nu \in L^2([0, +\infty), \mathbb{R}^n).\]

Consider the decomposition \(\nu(t) = (x(t), u^2(t))^T\), where \(x(t) \in \mathbb{R}^n\). By Theorem 1 it follows that \(\mathcal{M}(Ex(t)) = v^*(t)\) and hence \(v^* \in L^2([0, +\infty), \mathbb{R}^n)\). As \(g^*\) is a linear function of \(x, u, v\) (see Theorem 1) it follows that \(g^* \in L^2([0, +\infty), \mathbb{R}^k)\). Recalling that \(\hat{v}(t) = A_t v^*(t) + B_t g^*(t)\) we write: \(v^*(t) = v^*(\tau) + A_t \int_{\tau}^t v^*(s)ds + B_t \int_{\tau}^t g^*(s)ds\)
for \( \tau < t \). Since \( \int_\tau^t \|v^*(s)\|ds \leq (t-\tau)\frac{1}{2} \|v^*\|_{L^2([0,\infty), \mathbb{R}^n)} \) where \( \|v^*\|_{L^2([0,\infty), \mathbb{R}^n)} := \int_0^\infty \|v^*(s)\|_{\mathbb{R}^n}^2 ds \), it follows that:

\[
\|v^*(t) - v^*(\tau)\|_{\mathbb{R}^n} \leq (t-\tau)^\frac{1}{2} (\|A_1\|\|v\|_{L^2([0,\infty), \mathbb{R}^n)} + \|B_1\|\|g^*\|_{L^2([0,\infty), \mathbb{R}^k)}).
\]

Hence \( v^* \) is uniformly continuous. This and \( v^* \in L^2([0,\infty), \mathbb{R}^k) \) together with Barbalat’s lemma imply that \( \lim_{t \to \infty} v^*(t) = 0 \). Hence the optimal state trajectory \( v^* \) converges to zero, i.e. \( g^* \) is a stabilizing control for \( v_0 \), and thus \( v_0 = \mathcal{M}(z) \in \mathcal{V}_g \).

(ii) \( \implies \) (i) Assume now that \( v_0 = \mathcal{M}(z) \in \mathcal{V}_g \). Let us recall the construction of the ODE-LTI \( \mathcal{J}_g \). It then follows that

\[
T(\mathcal{V}_g) = \left[ \begin{array}{c}
I_l \\ 0 \\
0 \\ \ \ 0 
\end{array} \right],
\]

where \( l = \dim \mathcal{V}_g \). Define the map \( \Pi = \left[ \begin{array}{c} I_l \\ 0 \end{array} \right] \). Then it follows that for any \( v_0 \in \mathcal{V}_g \), and for any \( g \in L^1_{loc}([0,\infty), \mathbb{R}^k) \), \( (x^T, u^T)^T \) is the output of \( \mathcal{J} \) and \( t \mapsto v(t) \) is the state trajectory of \( \mathcal{J} \) starting from \( v_0 \) and driven by the input \( u \), if and only if \( (x^T, u^T)^T \) is the output of \( \mathcal{J} \) and \( t \mapsto \Pi(v)(t) \) is the state trajectory of \( \mathcal{J}_g \) starting from \( v_0 \) and driven by \( g \). For any initial state \( \hat{v}_0 \) of \( \mathcal{J}_g \) define now the cost function \( \mathcal{J}_t(\hat{v}_0, g) \) as

\[
\mathcal{J}_t(\hat{v}_0, g) = v^T(t)(E\hat{C}_s)^TQ_0E\hat{C}_sv(t) + \int_0^t v^T(s)Sv(s)ds,
\]

where \( \hat{C}_s \) is the matrix formed by the first \( n \) rows of \( C_g \). Notice that from the definition of \( C_g \) it follows that \( C_g = C_\Pi^+ \) (notice that \( \Pi^+ = T^{-1} \left[ \begin{array}{c} I_l \\ 0 \end{array} \right] \) ) and hence \( \hat{C}_s = C_s\Pi^+ \). Define now \( \mathcal{J}_\infty(\hat{v}_0, g) = \limsup_{t \to \infty} \mathcal{J}_t(\hat{v}_0, g) \).

Recall from (26) and (27) the definition of the cost functions \( \mathcal{J}_\infty \) and \( \mathcal{J}_t \). It is not hard to see that:

\[
\mathcal{J}_t(\Pi(v_0), g) = \mathcal{J}_t(v_0, g),
\]

\[
\mathcal{J}_\infty(\Pi(v_0), g) = \mathcal{J}_\infty(v_0, g),
\]

for any initial state \( v_0 \) of \( \mathcal{J} \) such that \( v_0 \in \mathcal{V}_g \).

Consider now the problem of minimizing \( \lim_{t \to +\infty} \mathcal{J}_t(\hat{v}_0, g) \). Let us apply the feedback transformation \( g = \hat{F}v + Uw \) to \( \mathcal{J}_g = (A_g, B_g, C_g, D_g) \), where \( \hat{F} \) and \( U \) are defined as follows. If \( D_1 \) is of full column rank, then \( U = (D_g^T S D_g)^{-1/2} \) and \( \hat{F} = -(D_g^T S D_g)^{-1} D_g^T S C_g \). If \( D_1 = 0 \), then \( \hat{F} = 0 \) and \( U = 1 \).

Recall the ODE-LTI (32) and the corresponding cost function \( \mathcal{J}_t(\hat{v}_0, w) \) from the proof of Lemma 2. By construction of \( \hat{F} \) and \( U \), \( \|S^{1/2}(C_g + D_g\hat{F})v(t) + D_gUw(t)\|^2 = v^T(t)(C_g + D_g\hat{F})^T S(C_g + D_g\hat{F})v(t) + w^T(t)w(t) \). Using these remarks, it is then easy to see that \( \mathcal{J}_t(\hat{v}_0, g) = \mathcal{J}_t(\hat{v}_0, w) \) for \( g = Fv + Uw \).

Consider now the problem of minimizing \( \lim_{t \to \infty} \mathcal{J}_t(\hat{v}_0, w) \). First, we assume that \( D_g = D_1 \) is full column rank. We apply [17, Theorem 3.7]. In the proof of Lemma 2 it was already shown that \( (A_g + B_g\hat{F}, B_gU) \) is stabilizable and \( (S^{1/2}(C_g + D_g\hat{F}), A_g + B_g\hat{F}) \) is observable. Let us now return to the minimization problem. Notice that (20) is equivalent to the algebraic Riccati equation described in [17, Theorem 3.7] for the problem of minimizing \( \lim_{t \to \infty} \mathcal{J}_t(\hat{v}_0, w) \). From Lemma 2 it follows that (20) has a unique positive definite solution \( P \), and \( A_g + B_g\hat{F} - B_gUU^TB_g^TP = A_g - B_gK \) is a stable matrix. From [17, Theorem 3.7], there exists \( w^* \in L^2([0,\infty), \mathbb{R}^k) \) such that \( \lim_{t \to \infty} \mathcal{J}_t(\hat{v}_0, w^*) \) is minimal and \( \hat{v}_0^* P \hat{v}_0^* = \lim_{t \to \infty} \mathcal{J}_t(\hat{v}_0, w^*) \). On the other hand, [17, Theorem 3.7] also implies that \( \hat{v}_0^* P \hat{v}_0^* = \lim_{t \to \infty} \inf_{w \in L^2_{loc}([0,\infty), \mathbb{R}^k)} \mathcal{J}_t(\hat{v}_0, w) \). Hence, \( g^* = Fv^* + Uw^* \) satisfies

\[
\mathcal{J}_\infty(\hat{v}_0, g^*) = \hat{v}_0^* P \hat{v}_0^* = \lim_{t \to \infty} \inf_{w \in L^1_{loc}([0,\infty), \mathbb{R}^k)} \mathcal{J}_t(\hat{v}_0, w),
\]

where \( \hat{v}^* = (A_g + B_g\hat{F})v^* + B_gUw^* \) and \( v^*(0) = \hat{v}_0 \). A routine computation reveals that \( (v^*, g^*) \) satisfies

\[
v^* = A_g v^* + B_g g^* \quad \text{and} \quad g^* = -Kv^* \quad \text{and} \quad v^*(0) = \hat{v}_0.
\]
Assume now that $D_I = D_g = 0$. Then $B_g = 0$ and $k = 1$. Moreover, in this case, the solution $P = P^T > 0$ of (20) is in fact the observability gramian of the ODE-LTI $\dot{v} = A_g v$, $y = S^{1/2} C_g v$. From the well-known properties of observability gramian it then follows that $\hat{v}_0^T P \hat{v}_0 = \int_0^\infty y^T(s) y(s) ds = \lim_{t \to \infty} \mathcal{F}_t(v_0, 0) = \mathcal{F}_\infty(v_0, 0)$. Since $B_g = 0$, $\mathcal{F}_t(v_0, 0) = \mathcal{F}_t(v_0, w) = \mathcal{F}_t(v_0, v)$ ($g = Fe + Uw = w$ if $D_g = 0$) for any $w \in L^1_{\text{loc}}([0, +\infty), \mathbb{R})$, $t \in [0, +\infty)$, hence $\hat{v}_0^T P \hat{v}_0 = \lim_{t \to \infty} \inf_{g \in L^1_{\text{loc}}([0, t], \mathbb{R}^k)} \mathcal{F}_t(v_0, g)$.

Hence, in both cases ($D_g = D_I$ is full rank or $D_I = 0$), from Theorem 7 it then follows that $(x^*, u^*)^T = C_g v^* + D_g g^*$ is a solution of the infinite horizon optimal control problem and that $(x^*, u^*)$ satisfies (21) and (23).

$(i) \implies (iii)$ If $(x^*, u^*) \in \mathcal{D}(\infty)$ is a solution of the infinite horizon optimal control problem, then, by definition, $+\infty > J_\infty(x^*, u^*) = J^* = \limsup_{t \to \infty} \inf_{(x, u) \in \mathcal{D}(t)} J_t(x, u)$.

$(iii) \implies (ii)$ From (28) it follows that the condition of $(iii)$ implies that there exists $M > 0$ such that for all $t > 0$,

$$\inf_{g \in L^1([0, t], \mathbb{R}^k)} \mathcal{F}_t(M(z), g) \leq M. \quad (34)$$

Recall that $S = \text{diag}(Q, R)$ and for each $g \in L^1_{\text{loc}}(I, \mathbb{R}^k)$, $[0, t] \subseteq I$, and initial state $v_0$, define

$$\mathcal{K}_t(v_0, g) = \int_0^t (Cv(s) + Dg(s))^T S(Cv(s) + Dg(s))ds \quad \dot{v} = Av + Bg, \quad v(0) = v_0.$$  

It then follows that $\mathcal{K}_t(v_0, g) \leq \mathcal{F}_t(v_0, g)$ for any $t > 0$ and $\mathcal{K}_t(v_0, g)$ is non-decreasing in $t$. Hence, (34) implies that

$$\forall t \in (0, +\infty) : \inf_{g \in L^1_{\text{loc}}([0, t], \mathbb{R}^k)} \mathcal{K}_t(v_0, g) < M.$$  

From classical linear theory [17, 27] it follows that if $H$ is the unique symmetric, positive semi-definite solution of the Riccati equation

$$\dot{H}(t) = A^T(t) H(t) + H(t) A(t) - K^T(t) (D(t)^T S D(t)) K(t) + C^T(t) C(t)$$
$$K(t) = (D(t)^T S D(t))^{-1} (B(t)^T H(t) + D(t)^T C(t)) \text{ and } H(0) = 0 \quad (35)$$

then $v_0^T H(t)v_0 = \inf_{g \in L^1_{\text{loc}}([0, t], \mathbb{R}^k)} \mathcal{K}_t(v_0, g)$. The latter may be easily seen applying the state feedback transformation $g = Fv + Uw$ with $F, U$ defined as in the proof of Theorem 5 and solve the resulting standard LQ control problem for the transformed system. It then follows that (35) is the differential Riccati equation which is associated with this problem. Note that the matrix $H(t)$ is symmetric and positive semi-definite, since $b^T H(t) b$ is monotonically non-decreasing for all $b$. Define the set

$$V = \{ v_0 \mid \sup_{t \in (0, +\infty)} v_0^T H(t)v_0 < +\infty \}.$$  

By assumption $(iii)$ it follows that $\mathcal{M}(z) \in V$. From [27, Theorem 10.13] it follows that $V_g \subseteq V$. It is also easy to see that $V$ is a linear space. We will show that $V = V_g$, and so $\mathcal{M}(z) \in V_g$ follows.

First, we will argue that $V$ is invariant with respect to $A_I$. To this end, consider $v_0 \in V$ and set $v_1 = e^{-A_I t_1} v_0$. For any $t_1 \in [0, +\infty)$ and any $g \in L^1_{\text{loc}}([0, t_1], \mathbb{R}^k)$, define $\hat{g} \in L^1_{\text{loc}}([0, t_1], \mathbb{R}^k)$ as $\hat{g}(s) = 0$, $s \leq t$ and $\hat{g}(s) = g(s - t)$ if

2 Indeed, recall the system $\mathcal{G}_g = (A_g, B_g, C_g, D_g)$ and the map II. Recall that $\mathcal{G}_g$ is stabilizable and hence by [27, Theorem 10.19] for every $v_0 \in V_g$ there exists an input $g \in L^1([0, +\infty), \mathbb{R}^k)$ such that $\int_0^{+\infty} ((C_g v(s) + D_g g(s))^T S(C_g v(s) + D_g g(s))ds < +\infty$ for $\dot{v} = A_g v + B_g g, \quad v(0) = \Pi(v_0)$. Consider the state trajectory $r(t)$ where $\dot{r} = Ar + Bg$, $r(0) = v_0$. It then follows that $\Pi(r) = r$ and hence $Cr + Dg = C_g v + D_g g$. Therefore, $v_0^T H(t_1) v_0 \leq \mathcal{K}_t(v_0, g) \leq \int_0^{+\infty} ((C_g v(s) + D_g g(s))^T S(C_g v(s) + D_g g(s))ds < +\infty$. 

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\( s > t \). Consider \( \dot{v} = A_tv + B_t\dot{g}, v(0) = v_1 \). It then follows \( v(t) = v_0 \) and hence

\[
\mathcal{H}_{t_1}(v_1, \dot{g}) = \mathcal{H}_{t_1}(v_0, \dot{g}) + \int_0^t (C_te^{A_ts}v_1)^T S(C_te^{A_ts}v_1)ds
\]

(37)

Since \( v_0 \in V \), it then follows that there exists \( \Gamma > 0 \) such that for any \( t_1 \) there exists \( g \in L^2([0, t_1], \mathbb{R}^k) \) such that \( v_0^T H(t_1)v_0 = \mathcal{H}_{t_1}(v_0, \dot{g}) < \Gamma \). Hence, from (37) it follows that \( v_0^T H(t_1 + t)v_1 = \inf_{\dot{g}} \mathcal{H}_{t_1+t}(v_1, \dot{g}) \leq \Gamma + \int_0^t (C_te^{A_ts}v_1)^T S(C_te^{A_ts}v_1)ds < +\infty \), and hence

\[
\sup_{t_1 \in (0, +\infty)} v_0^T H(t_1)v_1 \leq \sup_{t \in (0, +\infty)} v_0^T H(t_1 + t)v_1 < +\infty.
\]

In the last step we used that \( v_0^T H(t_1 + t_1)v_1 \geq v_0^T H(t_1)v_1 \) for all \( t, t_1 \in [0, +\infty) \). Hence, \( v_1 = e^{-A_tt}v_0 \in V \). Since \( V \) is a linear space, and \( t \) is arbitrary, it then follows that \( A Tv_0 = -\frac{d}{dt} e^{-A_tt}v_0 |_{t=0} \in V \). That is, \( V \) is \( A_t \) invariant.

Notice that the controllability subspace of \( \mathcal{S} \) is contained in \( V \) and that \( \text{im} B_t \subseteq \mathcal{S} \). Hence, \( \text{im} B_t \subseteq V \). Now, for any \( b \in V \), the function \( b^T H(t)b \) is monotonically non-decreasing in \( t \) and it is bounded, hence \( \lim_{t \to \infty} b^T H(t)b \) exists and it is finite. Notice for any \( b_1, b_2 \in V \), \( (b_1 + b_2)^T H(t)(b_1 + b_2) = b_1^T H(t)b_1 + 2b_1^T H(t)b_2 + b_2^T H(t)b_2 \) and as \( b_1 + b_2 \in V \), the limit on both sides exists and so the limit \( \lim_{t \to \infty} b_1^T H(t)b_2 \) exists. Consider now a basis \( b_1, \ldots, b_r \) of \( V \) and for any \( t \) define \( H_{i,j}(t) = b_i^T H(t)b_j \). It then follows that the matrix \( \hat{H}(t) = (H_{i,j}(t))_{i,j=1,\ldots,r} \) is positive semi-definite, symmetric and there exists a positive semi-definite matrix \( \hat{H} \) such that \( \hat{H} = \lim_{t \to \infty} \hat{H}(t) \). From (35) and \( A_tV \subseteq V \) it follows that \( A_t b_i \in V \) for all \( i = 1, \ldots, r \) and hence \( \lim_{t \to \infty} b_i^T H(t)b_j \) exist for all \( i, j = 1, \ldots, r \). From \( \text{im} B_t \subseteq V \) and the fact that for any \( x, z \in V \), the limit \( \lim_{t \to \infty} x^T H(t)z \) exists, it follows that the limits \( \lim_{t \to \infty} b_i^T H(t)x = \lim_{t \to \infty} x^T H(t)b_i \) exist for all \( x \in V \). Applying this remark to \( x = b_i \) and \( x = b_j \), it follows that \( \lim_{t \to \infty} b_i^T K(t)^TD_t^T S_D(t)K(t)b_j \) exists. Hence, for any \( i, j = 1, \ldots, r \), the limit of \( \hat{H}_{i,j}(t) = b_i^T H(t)b_j \) exists as \( t \to \infty \) and hence the limit \( \lim_{t \to \infty} \hat{H}(t) =: Z \) exists. Moreover, since \( \hat{H}(t) \) is symmetric and positive semi-definite, it follows that \( Z \) is symmetric and positive semi-definite.

We claim that \( Z \) is zero. To this end it is sufficient to show that \( \lim_{t \to \infty} b_i^T H(t)b = 0 \) for any \( b \in V \). Indeed, from this it follows that \( Z_{i,j} = \lim_{t \to \infty} 0.5((b_i + b_j)^T H(t)(b_i + b_j) - b_i^T H(t)b_i - b_j^T H(t)b_j) = 0 \) for any \( i, j = 1, \ldots, r \).

Now, fix \( b \in V \) and assume that \( c = \lim_{t \to \infty} b_i^T H(t)b \neq 0 \). Set \( h(t) = b_i^T H(t)b \). It then follows that \( \dot{h}(t) = b_i^T H(t)b \) and thus there exists \( T > 0 \) such that for all \( t > T \), \( h(t) > \frac{c}{2} > 0 \) (recall that \( H(t) \) is positive semi-definite). Hence, \( h(t) = h(0) + \int_0^t h(s)ds > \int_0^T h(s)ds > (t - T)\frac{c}{2} \). Hence, \( h(t) \) is not bounded, which contradicts to the assumption that \( b \in V \).

Hence, \( Z = 0 \) and thus, it follows that \( \hat{H} \) satisfies the algebraic Riccati equation

\[
0 = A_t^T \hat{H} + \hat{H}A_t - K^T (D_tD_t^T)K + \hat{C}_t^T S\hat{C}_t
K = (D_t^T S_D(t))^{-1}(\hat{B}_t^T \hat{H} + D_t^T \hat{C}_t)
\]

(38)

where \( \hat{A}_t, \hat{B}_t, \hat{C}_t \) are defined as follows: \( \hat{A}_t \) and \( \hat{C}_t \) are the matrix representations of the linear maps \( A_t \) and \( C_t \) restricted to \( V \). \( \hat{B}_t \) is the matrix representation of the map \( \mathbb{R}^k \ni g \mapsto B_tg \in V \) in the basis \( b_1, \ldots, b_r \) of \( V \) chosen as above. Note that for \( A_t \) and \( B_t \) to be well defined, we had to use the facts \( \text{im} B_t \subseteq V \) and \( A_tV \subseteq V \). Notice that \( D_t \) is injective as a linear map and recall from Remark 1 that \( C_tD_t + D_tg = 0 \) implies that \( p = 0, g = 0 \) and hence the largest output nulling subspace \( V(\Sigma) \) of the linear system \( \Sigma = (A_t, \hat{B}_t, \hat{C}_t, D_t) \) is zero. Then from [27, Theorem 10.19], \( V(\Sigma) \neq 0 \) and (38) it follows that \( \Sigma \) is stabilizable. Since \( \Sigma \) is just the restriction of \( \mathcal{S} \) to \( V \), it then follows that every state from \( V \) is stabilizable and hence \( V \neq \mathcal{V}_g \).

Let us now prove that the optimal solution satisfies (21), (23), \( u^* = K_f x^* \) and that it is a solution of (24) such that \( E x^* = z \). From the proof of (i) \( \rightarrow \) (ii) it follows that \( (x^*, u^*) \) from (21) is a solution of the infinite horizon optimal control problem for the initial state \( z \) and that it satisfies (23). Moreover, from (21) it follows that \( u^* = K_f x^* \).
Finally, all the solutions \((x, u)\) of (24) on \([0, +\infty)\) satisfy \((x^T, u^T)^T(s) = (C_g - D_g K)v(s)\) a.e., \(v(s) = (A_g - B_g K)v(s)\), \(v(s) = M_g(Ex(s))\), \(s \in [0, +\infty)\). Hence, \((x^*, u^*)\) is indeed a solution of (24) such that \(Ex^*(0) = z\) and any other solution \((x, u)\) of (24) with \(Ex(0) = z\) satisfies \(x = x^*, u = u^*\) a.e. ■

4 Numerical example

The spread of heat in a body can be described by the partial differential equation (PDE) of heat transfer,

\[
\frac{dV(t)}{dt} = AV(t) + Bu(t),
\]

(39)

where \(V(t), u(t)\) take values in the space \(H := L^2([-1, 1], \mathbb{R})\) and \(\mathbb{R}^{N_x}\) respectively, and \(AV = -c^2 \frac{d^2}{dx^2}v\) for all \(v \in H\) such that \(Av \in H, Bv = \sum_{k=1}^{N_u} \sin(k \pi x) v_k\) for all \(v = (v_1, \ldots, v_{N_u})^T \in \mathbb{R}^{N_u}\). We say that a pair \((V, u)\) solves (39) if \(u \in L^2_{loc}([0, +\infty), \mathbb{R}^{N_u})\), and \(V : [0, +\infty) \to H\) is a Frechet differentiable function such that \(AV(t)\) is defined, and (39) holds a.e., and \(V(t)(-1) = V(t)(1) = 0\). The PDE (39) is the simplest model for heat transfer. From a control perspective, \(u\) is the control input. We would like to find \(u\) with minimal energy that minimizes the energy of the heat distribution, i.e., \(J(V, u) = \int_0^\infty \|V(t)\|_H^2 + \|u(t)\|^2 dt\) is minimal. Here \(\|\|_H\) is the standard norm of the space \(H\), i.e. \(\|\phi\|_H^2 = \int_{-1}^1 \|\phi(v)\|^2 dv\).

To compute the optimal control PDEs are usually approximated by finite-dimensional models. Below we will do the same: we will present a DAE-LTI model which approximates (39), and, accordingly, the problem of minimizing \(J(V, u)\) of (39) onto a given finite-dimensional subspace. This, in turn, allows to find an exact minimizer \((V_{opt}, u_{opt})\) of \(J\). We will compare this minimizer against the control law obtained by solving Problem 2. Note, that for more complicated PDEs, the eigen-basis may not be available and, hence, it may be impossible to compute the exact minimizer of \(J\). However, the method based on approximating a PDE by a DAE-LTI does not rely upon the existence of this basis and applies in the general case. Our experiment shows that the trajectory of the DAE-LTI approximates well the optimal solution of (39), and the corresponding control law performs well when applied to (39). Note that this control law does not enforce the optimal solution of the DAE-LTI, when applied to the original model it enforces a near optimal solution. The practical significance of this is as follows. The DAE-LTI model and the corresponding instance of Problem 2 can be applied to PDEs which are more general than (39). However, the exact optimum of \(J(V, u)\) can be calculated only for (39), even small modifications of (39) to render it more realistic destroy this property. That is, DAE-LTIs could be a valuable tool for controlling systems described by PDEs. The reason we chose a model for which the control problem can be solved exactly, is that it allows us to compare the performance of DAE-LTI based control with the true optimal one. Otherwise, it is not clear how to evaluate the performance of DAE-LTI based controllers.

First we compute the optimal solution \(V_{opt}\) and \(u_{opt}\). To this end, consider the orthogonal basis \(\{\sin(\pi k v)\}_{k=1}^\infty\) in \(H\) and fix an integer \(N > 0\). We will follow the classical projection Galerkin method, and represent the coordinates of the solution \(V\) of (39) as a solution of an ODE-LTI. To this end, let \(A_v\) be a diagonal matrix whose \(i\)th diagonal element is \(-c^2 i^2 \pi^2\), \(i = 1, \ldots, N\), and let \(B_v = I_{N_x}\). Then for any \(u \in L^2_{loc}([0, +\infty), \mathbb{R}^{N_u})\), \(z(t) = (z_1(t), \ldots, z_N(t))^T\) is a solution of

\[
\dot{z} = A_v z + B_v u,
\]

(40)

if and only if \((V, u), V(t)(v) = \sum_{k=1}^N z_k(t) \sin(k \pi v)\) is a solution of (39) and \(J(V, u) = J_e(z, u) \overset{def}{=} \int_0^\infty \|z(t)\|^2 + \|u(t)\|^2 dt\). Hence, if we want to minimize \(J(V, f)\) with the additional restriction that \(V(0)(v) = \lambda \sin(\pi n v)\) for some \(n \leq N\) and \(\lambda > 0\), then it is equivalent to minimizing \(J_e(z, u)\) subject to (40) for \(z(0) = \lambda \epsilon_n\). The latter problem is a standard LQ control problem. The optimal trajectory \((z_{opt}, u_{opt})\) of (40) uniquely determines the optimal solution \(V_{opt}, u_{opt}\) of (39), and \(u_{opt} = K_{opt} z_{opt}\) for some matrix \(K_{opt}\) (equivalently, \(u_{opt} = K_{opt} V_{opt}\) for a suitable bounded operator \(K_{opt}\)). In particular, for \(\lambda = 10, n = 34, N = 40, N_u = 35\) the optimal cost is \(J_e(z_{opt}, u_{opt}) = J(V_{opt}, u_{opt}) = 3.94\). The approach presented above is a particular instance of the classical Galerkin projection method, applied to the basis \(\{\sin(\pi n v)\}_{n=1}^\infty\).

The approach above relies heavily on the fact that the functions \(\{\sin(\pi n v)\}_{n=1}^\infty\) are eigenfunctions of \(A\), and hence cannot easily be generalized to other PDEs. However, it is difficult to find eigenfunctions for \(A\) if it is slightly modified
so that it is more realistic. However, if we choose just any basis and we apply the classical Galerkin projection method, then the resulting ODE-LTI will give a poor approximation of solutions of (39). For this reason, we use the approach of [35] for constructing a DAE-LTIs whose solutions approximate the solutions of (39). Consider the (non-orthogonal) basis \( \{\phi_k\}_{k=0}^{\infty} \) of \( H \), where \( \phi_k = P_{k+1} - P_k \), \( k \geq 1 \), with \( P_i \) denoting the \( i \)th Legendre polynomial, \( i = 0, 1, \ldots \). In practice, this basis is often chosen due to its numerical properties. Define the linear map \( P_N, P_N^+ \), and the matrices \( M_N, \hat{M}_N, A_N \) as

\[
\forall \psi \in H : P_N(\psi) = \left[ \langle \psi, \phi_1 \rangle, \ldots, \langle \psi, \phi_N \rangle \right]^T
\]

\[
\forall z = (z_1, \ldots, z_N)^T \in \mathbb{R}^N : P_N^+(z) = \sum_{k=1}^{N} z_k \phi_k
\]

\[
\hat{M}_N = (\langle \phi_i, \phi_j \rangle)_{i,j=1}^{N}, \quad A_N = (\langle A\phi_i, \phi_j \rangle)_{i,j=1}^{N}
\]

\[
M_N = \Lambda_N (\langle \phi_i, \phi_j \rangle)_{i,j=1}^{N}, \quad \Lambda_N = (\frac{2i+1}{2}\delta_{i,j})_{i,j=1}^{N}
\]

where \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( H \) and \( \delta_{i,j} \) is the Kronecker delta symbol for all \( i, j = 1, 2, \ldots \). Intuitively, if \( \phi \in H \), then \( a = \hat{M}_N^{-1} P_N \phi \) is the vector of the first \( N \) coordinates of \( \phi \) w.r.t. \( \{\phi_k\}_{k=1}^{\infty} \). From [35] it follows that if \( (V, u) \) is a solution of (39), then with \( a = \hat{M}_N^{-1} P_N V \), and \( e_m = P_N A e \), \( e = V - P_N^+ P_N V \), and \( x = (a^T, e_m^T)^T \), the pair \((x, u)\) is a solution of (1), where

\[
E = \left[ M_N, 0 \right], \quad A = \left[ \Lambda_N A_N, \Lambda_N \right],
\]

\[
B = \Lambda_N \left[ P_N(\sin(\pi v)), \ldots, P_N(\sin(\pi N u v)) \right]^T.
\]

Since according to [11, Example 7.2, p. 121], \( \langle \phi_i, \phi_j \rangle = \frac{(2i+1)^2}{(2i-1)(2i+3)} \delta_{i,j} - \frac{2}{2i-3} \delta_{i+2,j} - \frac{2}{2i-1} \delta_{i-2,j}, \) and \( \langle A\phi_i, \phi_j \rangle = -2c(2i+1) \delta_{i,j} \) for all \( i, j = 1, \ldots, N \), it follows that \( E, A \) can easily be computed, while \( B \) can be computed using numerical integration. The matrices \( M_N, \Lambda_N, A, B, E, \hat{M}_N, A_N \) are presented in Appendix A for \( N = 40, Nu = 35, \mu = 0.01 \) and \( c = \frac{1}{30} \). The corresponding matrices and the code for generating them can be downloaded from http://sites.google.com/site/mihalypetreczky/. Intuitively, the component \( a(t) \) of \( x(t) \) represents the first \( N \) components of \( V(t) \) w.r.t \( \{\phi_k\}_{k=1}^{\infty} \), and \( e_m \) is a linear function of the error \( e \) of approximating \( V \) by its projection to the first \( N \) basis vectors. The smaller \( e \) is, the closer the behavior of (1) is to that of (39). Conversely, if \((x, u)\) is a solution of (1), and \( a \) is the vector of the first \( N \) components of \( x \), then we can define \( V(t) = P_N^+ a(t) \) and view \((V, u)\) as an approximation of a solution of (39). Note that, in general, \((V, u)\) obtained in this way will not be a true solution of (39). Consider Problem 2, with

\[
Q = \text{diag}(M_N, \mu I_N), \quad R = I_{N_2}, \quad Q_0 = 0
\]

with \( \mu > 0 \) and \((E, A, B)\) defined above. The choice of \( Q \) is motivated by the following observation. If \((x, u)\) is the solution of (1) arising from a solution \((V, u)\) of (39), and we define \( V_{\text{approx}}(t) = P_N^+ a(t) \), then \( J(V_{\text{approx}}, u) = J_\infty(x, u) - \mu \int_0^\infty \|e_m^2(t)\|_2 dt \). Hence, if \( e = V - V_{\text{approx}} \) and \( e_m \) are small, i.e. \( V_{\text{approx}} \) is a good approximation of \( V \), then \( J(x, u) \) is close to \( J(V_{\text{approx}}, u) \), and hence to \( J(V, u) \), and it is reasonable to minimize \( J_\infty(x, u) \) instead of \( J(V, u) \).

The results of [13–16, 22–25] do not apply to this instance of Problem 2. If it was the case, then there would exist a feedback \( u = Kx = K_1a + K_2e_m \), with the decomposition \( x = (a^T, e_m^T)^T \), such that all the trajectories of the closed-loop system are at least stable. However, the closed-loop system reads as follows: \( \dot{x} = (\Lambda_N A_N + BK_1)a + (\Lambda_N + BK_2)e_m \), and \( e_m \) can be any square-integrable function. Hence, we can always choose \( e_m \) so that it does not converge to zero as \( t \to +\infty \). This is not surprising, since \( e_m \) is related to the error of approximating (39) by a finite-dimensionsal system, and this error cannot be controlled.

We computed a stabilizable associated ODE-LTI, solved (20) and computed the optimal trajectory for the case: \( N = 40, Nu = 35, \mu = 0.01 \) and \( c = \frac{1}{30} \). The corresponding matrices and the code for generating them and for solving the optimal control problem can be downloaded from http://sites.google.com/site/mihalypetreczky/, and they are also presented in Appendix A. From the specific initial condition \( \dot{x}(0) = \Lambda_N P_N v_{\text{opt}}(0) \), we computed the optimal trajectory \((x_*, u_*)\), and the optimal cost \( J_\infty(x_*, u_*) = 3.89 \). As mentioned earlier, the solution \((V_{\text{opt}}, u_{\text{opt}})\)
of (39) which satisfies $V_{opt}(0)(v) = \lambda \sin(\pi n v)$, $\lambda = 10$, $n = 34$ can be computed by using the eigen-basis of $A$, see [19] for more details. The true optimal value is $J(V_{opt}, u_{opt}) = 3.94$, which is close to $J_{opt}(x_t, u_t)$. On Figure 1 we plotted the norm of the difference between $V_{t}(t) = (P_N^\ast a_t(t))$ and $V_{opt}(t)$, where $a_t(t)$ is the vector formed by the first $N$ components of $x_t$.

Recall that the optimal solution $(x^\ast, u^\ast)$ satisfies the feedback $u^\ast = K_f Ex^\ast$ for some matrix $K_f$. Define now $K_f = K_f M_N^\ast P_N = K_f \Lambda_N P_N$, and consider the feedback law $u = K_f V$ for (39). For a fixed initial value $V(0)$ the feedback $u = K_f V$ yields a unique solution $(V, u)$ of (39), and $(V, u)$ yields a solution $(x, u)$ of (1) which satisfies $u = K_f E x$. That is, those solutions of the closed-loop DAE-LTI (1) with $u = K_f E x$ arise from the solutions of the closed-loop system (39) with the control law $u = K_f V$. While the closed-loop DAE-LTI will have many solutions which minimize the closed-loop PDE (39) with $u = K_f V$ will have at most one. We computed a solution $(V_{sim}, u_{sim})$, $u_{sim} = K_f V_{sim}$ of (39) on the interval $[0, T]$, and we used the fact that $V_{sim}(0) = 0$. The cost $J(V_{sim}, u_{sim})$ was approximated by $J_{T}(V_{sim}, u_{sim}) = \int_0^T \|V_{sim}(t)\|^2_H + \|u_{sim}(t)\|^2_H dt$, yielding 3.96 which is close to $J_{opt}(V_{opt}, u_{opt})$.

Note that by increasing $T$, the value of $J_{T}(V_{sim}, u_{sim})$ did not change significantly. Furthermore, the difference between $V_{sim}(t) - V_{opt}(t)$ approaches zero as $t \to T$ and and $\max_{t \in [0, T]} |V_{sim}(t, v) - V_{opt}(t, v)|^2_H dv = 0.0015$. The corresponding plot is presented in Figure 2. Note that $V_{t} \neq V_{sim}$, since $(x_t, u_t)$ need not be a true solution of (39) while $(V_{sim}, u_{sim})$ is so. In fact, even the initial values are different: $V_{t}(0) \neq V_{opt}(0) = V_{sim}(0)$. This is not surprising, since $V_{t}(0) = (P_N^\ast a_t(0)$. The Matlab code for computing $V_{opt}$, $u_{opt}$, $V_{t}$, $(x_t, u_t)$, $(V_{sim}, u_{sim})$, the corresponding cost function and generating the plots can be downloaded from http://sites.google.com/site/mihalypetreczky/.

Note that we could have tried to approximate solutions of (39) using classical Galerkin projection method w.r.t. \{$\phi_k$\}$_{k=1}^\infty$. Then the problem of minimizing $J(V, u)$ is approximated by a classical LQ control problem for an ODE-LTI. However, the estimate of the optimal value of $J(V, u)$ and the performance of the corresponding controller obtained for the ODE-LTI are much worse than the one described in the previous paragraphs. More precisely, in this case we approximate the first $N$ coordinates the solution $V(t)$ of (39) w.r.t. \{$\phi_k$\}$_{k=1}^\infty$ by the solution of the ODE-LTI $\dot{a} = M_N^{-1} A_N a + M_N^{-1} B_u$, and we approximate $J(V, u)$ by $J_g(a, u) = \int_0^T a^\ast(t) M_N a(t) dt + u^\ast(t) u(t) dt$.

Instead of minimizing $J(V, u)$ we then minimize $J_g(a, u)$. The latter is a classical LQ problem. If $(a_t, u_t)$ is the minimal value of $J_g(a, u)$ subject to $a = M_N^{-1} A_N a + M_N^{-1} B_u$, then $(V_{g}, u_{g})$, $V_{g}(t) = P_N a_t$, can be viewed as an approximation of a solution of (39), and $J(V_{g}, u_{g}) = J_g(a_t, u_t)$. For $a(0) = P_N V_{opt}(0)$, we have estimated $J(V_{g}, u_{g}) = 6.13$, which is much higher than the true value $J(V_{opt}, u_{opt}) = 3.94$. The plot of $\|V_{g}(t) - V_{opt}(t)\|^2_H$ and together with $\|V_{g}(t) - V_{opt}(t)\|^2_H$ is shown on Figure 3. It is clear that $V_{g}$ is much further from $V_{opt}$ than $V_{t}$ and it converges to it much slower. When optimizing $J_g(q, u)$ we also obtain a feedback law $u = K_g a$. If we can interpret the feedback $u = K_g a$ as a feedback $u = K_g V$, $K_g = K_g M_N^{-1} P_N V$, and apply it to (39). The resulting solution $(V_{sim, g}, u_{sim, g})$ of the closed-loop system is far from the optimal $V_{opt}$ for $V_{sim, g}(0) = V_{opt}(0)$, and it converges slower to $V_{opt}$ than $V_{t}$ obtained by applying the feedback $u = K_f V$ computed by using DAE-LTI, see Figure 4. In fact, $J_{T}(V_{sim, g}, u_{sim, g}) = 5.55$, where $J_{T}(V_{sim, g}, u_{sim, g}) = \int_0^T \|V_{sim, g}(t)\|^2_H + \|u_{sim, g}(t)\|^2_H dt \leq J(V_{sim, g}, u_{sim, g})$. Hence, the value of $J(V, (sim, g), u_{sim, g})$ is much larger than the optimum $J(V_{opt}, u_{opt})$. This is not surprising, the approximation error, hence the resulting model is much less accurate. This example thus illustrates the advantages of using a DAE-LTI model over the traditional Galerkin projection method.

In order to compute $(V_{sim, g}, u_{sim, g})$ we used (40) in a similar manner as it was done to compute $(V_{sim}, u_{sim})$. That is, we used the fact that $V_{opt}(0)(v) = \lambda \sin(\pi n v)$, then any solution $(V, u)$ of (39) such that $V(0) = V_{opt}(0)$ satisfies $V(t)(v) = \sum_{k=1}^{\infty} z_k(t) \sin(\pi v)$, where $(z, u)$ satisfies (40). Hence, the feedback law $u = K_g V = K_g M_N^{-1} P_N V$ is equivalent to a feedback law $u = K_{1,g} z$ for (40), where the $j$th column of $K_{1,g}$ equals $K_g M_N^{-1} P_N \sin(\pi j v)$, $j = 1, \ldots, N$. We simulated (40) on $[0, T]$, with $u = K_{1,g} z$ and the initial state being $z(0) = \lambda e_n$, where $e_n$ is the nth
Fig. 1. Difference \( e_{\text{sol}}(t) = \| V_s(t) - V_{\text{opt}}(t) \|_H^2 \, dv \) between the true optimum and the optimal solution of (1).

Fig. 2. Difference \( e_{\text{sim}}(t) = \| V_{\text{sim}}(t) - V_{\text{opt}}(t) \|_H^2 \, dv \) between the true optimal solution and the solution obtained by applying the feedback computed from the DAE-LTI.

Fig. 3. Comparison with DAE-LTI: the difference \( e_{g}(t) = \| V_g(t) - V_{\text{opt}}(t) \|_H^2 \, dv \) between the true optimum and the optimal solution of the ODE-LTI obtained via Galerkin method, and \( e_{\text{sol}}(t) = \| V_{\ast}(t) - V_{\text{opt}}(t) \|_H^2 \, dv \).

The code for computing \( a_g, u_g, V_g, V_{\text{sim},g}, u_{\text{sim},g} \), the corresponding values of the cost functions and generating the plots can be downloaded from [http://sites.google.com/site/mihalypetreczky/](http://sites.google.com/site/mihalypetreczky/).

5 Conclusions

We have presented a framework for representing solutions of non-regular DAE-LTIs as outputs of ODE-LTIs and for solving an LQ infinite horizon optimal control problem for non-regular DAE-LTIs. The solution concept we adopted for DAE-LTIs allows non-differentiable solutions but excludes distributions. We have shown that ODE-LTIs representing solutions of the same DAE-LTI are feedback equivalent and minimal. Moreover, we presented necessary
and sufficient conditions for existence of a solution of the LQ infinite horizon control problem and an algorithm for computing the solution. The solution can be generated by adding an algebraic constraint to the original DAE-LTI and hence it can be interpreted as a result applying a controller in the behavioral sense [30].

References

A Matrices of the numerical example


The matrices $AN$, $MN$, $ΛN$, $\hat{MN}$:

$$
\begin{bmatrix}
\end{bmatrix}
$$
The matrices of the DAE-LTI:

\[
\begin{bmatrix}
\ldots&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
\]
The matrices of the associated ODE-LTI system are:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Stabilizable associated ODE-LTI

\[
\begin{bmatrix}
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{bmatrix}
\]