REALIZATION THEORY OF NASH SYSTEMS∗

JANA NEMCOVÁ†, MIHÁLY PETRECZKY‡, AND JAN H. VAN SCHUPPEN§

Abstract. This paper deals with realization theory of so-called Nash systems, i.e. nonlinear systems the right-hand sides of which are defined by Nash functions. A Nash function is a semi-algebraic analytic function. The class of Nash systems is an extension of the class of polynomial and rational systems and it is a subclass of analytic nonlinear systems. Nash systems occur in many applications, including systems biology.

Formulation of the realization problem for Nash systems and a partial solution to it are presented. More precisely, necessary and sufficient conditions for realizability of a response map by a Nash system are provided. The concepts of semi-algebraic observability and semi-algebraic reachability are formulated and their relationship with minimality is explained. In addition to their importance for systems theory, the obtained results are expected to contribute to system identification and model reduction of Nash systems.

Key words. Nash systems, realization theory, reachability, observability

AMS subject classifications. 93B15, 93C10, 93B25, 14P20

Realization theory is one of the central topics of system theory. It serves as a theoretical foundation for model reduction, system identification and filtering/observer design. Its aim is to answer the following questions:

(1) Under which conditions is it possible to construct a (preferably minimal) system of a certain class generating the specified input/output behaviour?

(2) How to characterize minimal systems of a certain class which generate the specified input/output behaviour?

In this paper we investigate realization theory of the class of Nash systems. Nash systems are continuous-time dynamical systems with Nash submanifolds as state-spaces and such that the right-hand sides of the differential equations which determine their dynamics, and the output functions are Nash functions. By a Nash function we mean an analytic function satisfying an algebraic equation. A Nash submanifold of $\mathbb{R}^n$ is a smooth manifold which is defined by polynomial equalities and inequalities. Nash functions and Nash manifolds are named after John Nash, who used them in his seminal paper on embedding of manifolds in euclidean space [24].

Our approach to realization theory for Nash systems is a continuation of the approach followed by [17] for nonlinear systems, [36, 4] for polynomial systems and the one in [41, 3, 28, 29] for rational systems. It is closely related to the more recent work [6, 5], although we consider a different system class. We introduce the framework of Nash systems and the concepts of semi-algebraic reachability and semi-algebraic observability. We link semi-algebraically reachable, semi-algebraically observable, and minimal Nash realizations. We expect that the results on realization theory for Nash systems will be useful for system identification, model reduction, filtering and control design of Nash systems.

∗This work was partially supported by the ITEA project Twins 05004, NWO project 613.000.442 and GACR project 13-16764P.
†Institute of Chemical Technology, Prague, Department of Mathematics, Technická 5, 166 28 Prague 6, Czech republic (Jana.Nemcova@vscht.cz).
‡Ecole des Mines de Douai, Department of Computer Science and Automatic Control, F-59500 Douai, France. mihaly.petreczky@mines-douai.fr.
§Centrum Wiskunde & Informatica (CWI), P.O. Box 94079, 1090 GB Amsterdam, The Netherlands (jan.h.van.schuppen@gmail.com).
To the best of our knowledge, both the class of Nash systems and the presented results on realization theory are new. This paper is based on the Ph.D. thesis of one of the authors, see [26]. All the major proofs and results of the paper have appeared in [26]. The results of the paper were also announced in [27]. The main difference between [27] and the current paper is that the latter provides all the detailed proofs. In addition, with respect to [27] the presentation of the current paper is more detailed and certain technical concepts, such as semi-algebraic observability of Nash systems, have been further refined.

The idea of studying semi-algebraic systems, i.e. systems described by polynomial equalities and inequalities, is quite well-established in the hybrid systems community, see [23, 22, 31]. In [30] realization theory of discrete-time semi-algebraic hybrid systems is investigated. Since the class of systems in [30] is different from the class of Nash systems considered in this paper, the results of this paper neither imply nor are implied by those of [30]. For semi-algebraic hybrid systems, problems related to realization theory are investigated in [10, 11, 12, 13, 21]. For polynomial and rational systems, realization theory, reachability and observability are treated in [1, 2, 3, 4, 41, 28, 29].

The study of Nash systems is motivated in Section 1. The necessary background on real algebraic geometry is presented in Section 2. In Section 3 the class of Nash systems is introduced. The realization problem for the class of Nash systems is formulated in Section 4. In addition, Section 4 presents the notions of minimality, semi-algebraic reachability and semi-algebraic observability for Nash systems. The results on the existence and the minimality of Nash realizations are presented in Section 5. The proofs are presented in Section 6 and Section 7. The appendix contains the proofs of a number of auxiliary results.

1. Motivation. The motivation for studying Nash systems can be summarized as follows:

- Nash systems are both computable and general.
- Nash systems are universal approximators.
- Nash systems are relevant for systems biology.

Let us elaborate on each of these aspects.

Generality and computability. The class of Nash systems lies between polynomial/rational systems and analytic systems. While more general than the former, it still allows a constructive description by means of finitely many polynomial equalities and inequalities. Hence, it might still be possible to derive computational methods for control and analysis of Nash systems.

Universal approximation property. It is well-known that (piecewise-)polynomial systems can be used to approximate the behavior of nonlinear systems, see [19]. Since polynomial systems are Nash, it follows that Nash systems might be used to approximate general nonlinear systems.

Relevance for systems biology. Polynomial and rational systems, and thus Nash systems, are commonly used in systems biology to model metabolic, signaling, and genetic networks. In [32, 33] Savageau proposes an alternative (so-called power-law) framework, also known as Biochemical Systems Theory, for modeling metabolic and gene-regulatory networks by dynamical systems which still belong to the class of Nash systems, but are more general than polynomial and rational systems. In this framework, see also [34, 39], all processes are modeled by products of power-law functions, i.e. sums of products $x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$ of state variables $x_1, \ldots, x_n$ taken to rational powers ($q_i, i = 1, \ldots, n$ are rational numbers). Such functions are a special
case of Nash functions. Note that the values of rational exponents (kinetic orders) in power-law systems can be related to parameters of different rate laws such as for example Michaelis-Menten kinetics, see [39]. The role of power-law systems in modeling of signal transduction pathways is discussed in [37]. For further examples and details on application of power-law, and thus Nash, systems in biology see [40, 18].

There are several frameworks for modeling metabolic networks. Some of them (including power-law framework) are compared in [14]. The tendency modeling framework which extends the power-law framework combines mass-action and power-law kinetics into tendency kinetics, [38]. It also leads to Nash systems.

Further, considering Nash submanifolds as the state-spaces allows a natural implementation of possible conservation laws and restrictions on the state variables. For example, one can capture the assumption on positivity of state variables which represent the concentrations of reactants of a chemical system by defining the state-space as the positive orthant of $\mathbb{R}^n$ which is a Nash submanifold.

2. Real algebraic geometry. In this paper we follow the notation and terminology of [42, 20, 8] on commutative algebra and real algebraic geometry. However, we do not state all results in their full generality (over arbitrary real closed fields), we work only over the field $\mathbb{R}$ of real numbers.

First, let us recall the notion of transcendence degree. Let $A$ be an integral domain and let $\mathbb{R}$ be a sub-ring of $A$. Recall from [42] the notion of quotient field (field of fractions) $Q_A$ of $A$. Then the transcendence degree of $A$ over $\mathbb{R}$, denoted in this paper by $\text{trdeg} A$, is defined as the transcendence degree over $\mathbb{R}$ of the field of fractions $Q_A$ of $A$. The transcendence degree of $Q_A$ (and hence of $A$) equals the greatest number of algebraically independent elements of $F$ over $\mathbb{R}$. In accordance with the terminology of [42], if $S, R$ are integral domains, $S$ is a sub-ring of $R$, then we say that $R$ is algebraic over $S$, if the field of fractions $Q_R$ of $R$ is an algebraic extension of the field of fractions $Q_S$ of $S$. This is equivalent to saying that for any element $r \in R$ there exists a polynomial $0 \neq Q \in S[X]$ such that $Q(r) = 0$. If $\text{trdeg} S = d$ and $x_1, \ldots, x_d$ is a finite transcendence basis of $S$, then $r \in R$ is algebraic over $S$ if and only if there exists a polynomial in $d + 1$ variables $P \in \mathbb{R}[X_0, \ldots, X_d]$ such that $P \neq 0$, $P(x_0, x_1, \ldots, x_d) \neq 0$ and $P(r, x_1, \ldots, x_d) = 0$. 

A subset $S \subseteq \mathbb{R}^n$ is called semi-algebraic if it is a set of points of $\mathbb{R}^n$ which satisfy finitely many polynomial equalities and inequalities, or if it is a finite union of such sets.

We say that a semi-algebraic subset $S \subseteq \mathbb{R}^n$ is semi-algebraically connected if it cannot be written as a union of two disjoint closed semi-algebraic sets in $S$. In [8] it is shown that semi-algebraically connected sets are also connected.

Let $S, S' \subseteq \mathbb{R}^n$ be semi-algebraic sets. A function $f : S \to S'$ is a semi-algebraic function if its graph is a semi-algebraic set in $\mathbb{R}^{n+m}$.

In the sequel we will need the following technical lemma on the existence of a partition of a semi-algebraic set.

Lemma 2.1 ([8]). Let $S \subseteq \mathbb{R}^n$ be a semi-algebraic set and let $f : S \to \mathbb{R}$ be a semi-algebraic function. There exist an integer $m > 0$ and semi-algebraic subsets $S_1, \ldots, S_m$ of $S$ and polynomials $g_i \in \mathbb{R}[X_1, \ldots, X_n, Y]$, $i = 1, \ldots, m$ such that $S = \bigcup_{i=1}^m S_i$, $S_i \cap S_j = \emptyset$ for all $i \neq j \in \{1, \ldots, m\}$ and such that for all $x \in S_i$, $g_i(x, f(x))$ is not identically zero and $g_i(x, f(x)) = 0$.

Next, we define the notion of Nash function and Nash manifold.

Definition 2.2. A Nash function on an open semi-algebraic set $S \subseteq \mathbb{R}^n$ is an analytic and semi-algebraic function from $S$ to $\mathbb{R}$. We denote the ring of Nash
functions on $S$ by $\mathcal{N}(S)$.

We say that a function $f : S \to \mathbb{R}^k$ is a Nash function, if the coordinate functions of $f$ are Nash functions, i.e. if for some $f_1, \ldots, f_k \in \mathcal{N}(S)$ it holds that $\forall x \in S : f(x) = (f_1(x), \ldots, f_k(x))$.

If $U$ and $V$ are open semi-algebraic subsets of $\mathbb{R}^n$ and $f : U \to V$ is a Nash function (i.e. $f$ is a Nash function if viewed as a map $f : U \to \mathbb{R}^n$) and the inverse $f^{-1}$ exists and it is a Nash function, then $f$ is called a Nash diffeomorphism.

**Definition 2.3.** A Nash submanifold $X$ of $\mathbb{R}^n$ is a semi-algebraic set which is also an analytic manifold. More precisely, $X$ is a Nash submanifold of $\mathbb{R}^n$ of dimension $d$, if the following holds. For every point $x \in X$ there exists

1. an open semi-algebraic set $\Omega_x \subseteq \mathbb{R}^n$ such that $0 \in \Omega_x$,
2. an open semi-algebraic set $\Omega'_x \subseteq \mathbb{R}^n$ such that $x \in \Omega'_x$, and
3. a Nash diffeomorphism $\varphi : \Omega_x \to \Omega'_x$ such that $\varphi(0) = x$ and $\varphi((\mathbb{R}^d \times \{0\}) \cap \Omega_x) = X \cap \Omega'_x$, where $\mathbb{R}^d \times \{0\} = \{(x_1, \ldots, x_d, 0, \ldots, 0) \in \mathbb{R}^n \mid (x_1, \ldots, x_d) \in \mathbb{R}^d\}$.

The sets $\Omega_x$ will be called coordinate charts and the maps $\varphi$ will be called coordinate maps. We will denote the dimension of $X$ by $\dim(X)$. A map $f : X \to \mathbb{R}^k$ is said to be a Nash function on $X$, if for every $x \in X$, the map $g : \Omega_x \ni (x_1, \ldots, x_n) \mapsto f(\varphi(x_1, \ldots, x_d, 0, \ldots, 0))$ is a Nash function. We denote the set of Nash functions of the form $f : X \to \mathbb{R}^k$ is said to be a Nash function, if every coordinate function of $f$ is a Nash function. If $M, N$ are two Nash manifolds, $N \subseteq \mathbb{R}^k$, then a map $f : M \to N$ is said to be Nash function, if $f$ viewed as a map $f : M \to \mathbb{R}^k$ is a Nash function. The Nash function $f : M \to N$ is said to be a Nash diffeomorphism, if its inverse $f^{-1} : N \to M$ exists and it is a Nash map. We will view each Nash manifold $M \subseteq \mathbb{R}^n$ as a topological space, with topology inherited from $\mathbb{R}^n$; we will say that a subset $U$ of $M$ is open if there exists an open subset $V$ of $\mathbb{R}^n$ such that $V \cap M = U$.

In fact, the definition of Nash manifolds implies that Nash manifolds are embedded analytic sub-manifolds of some Euclidian space (see [9] for the background material on smooth and analytic manifolds). The topology described above is just the corresponding topology for embedded submanifolds. If Nash manifolds are viewed as analytic embedded submanifolds of some Euclidian space, then Nash maps defined on them are analytic functions on manifolds.

Further, the Zariski closure of a Nash submanifold has, by [8, Proposition 8.4.1], the following property:

**Proposition 2.4.** Let $X \subseteq \mathbb{R}^n$ be a semi-algebraically connected Nash submanifold of dimension $d$. Then $\text{Z-cl}(X)$ is an irreducible real affine variety of dimension $d$.

Recall that the dimension of an affine variety equals the Krull dimension of the ring of polynomial functions of this variety. If $X$ is as in Proposition 2.4, then the ring of polynomial functions on $\text{Z-cl}(X)$ equals $\mathbb{R}[X_1, \ldots, X_n]/I(X)$, where $I(X) = \{ P \in \mathbb{R}[X_1, \ldots, X_n] \mid P|_X = 0 \}$ is the ideal of polynomials vanishing on $X$. We will call $\mathbb{R}[X_1, \ldots, X_n]/I(X)$ the ring of polynomials on $X$ and denote by $R[X]$. If $X$ is connected, then, by Proposition 2.4, $\text{Z-cl}(X)$ is an irreducible affine variety and thus $R[X]$ is an integral domain. Hence, according to [20], the Krull dimension of $R[X]$ equals its transcendence degree.

**Corollary 2.5.** If $X \subseteq \mathbb{R}^n$ is a semi-algebraically connected Nash submanifold of dimension $d$, then $\text{trdeg} R[X] = d$.

Further, we will need the following important property of Nash functions on Nash
manifolds.

**Proposition 2.6.** [8, Proposition 8.18] Let $X \subseteq \mathbb{R}^n$ be a Nash submanifold which is semi-algebraically connected, let $U$ be a non-empty open subset of $X$, and let $f : X \to \mathbb{R}$ be a Nash function. If $f|_U = 0$ then $f = 0$.

The proposition above implies that if two Nash functions are equal on an open neighborhood, then they are equal everywhere. Proposition 2.6 also implies the following.

**Proposition 2.7.** The ring $\mathcal{N}(X)$ of Nash functions defined on a semi-algebraically connected Nash manifold $X$ is an integral domain.

Indeed, assume $f_1 f_2 = 0$ for some $f_1, f_2 \in \mathcal{N}(X)$ and $f_1(z) \neq 0$ for some $z \in X$. Because $f_1$ is Nash and thus continuous, the set $U = \{x \in X \mid f_1(x) \neq 0\}$ is open. Further, since $z \in U$, $U$ is non-empty. Then, $f_1 f_2 = 0$ implies that $f_2|_U = 0$, and hence by Proposition 2.6, $f_2 = 0$.

Finally, we will need the following corollary of Lemma 2.1.

**Proposition 2.8.** Let $X$ be a Nash submanifold of $\mathbb{R}^n$ which is semi-algebraically connected. Let $f_1, \ldots, f_k \in \mathcal{N}(X)$ and $g \in \mathcal{N}(S)$ where $S$ is a Nash submanifold of $\mathbb{R}^r$. Assume that for all $x \in X$, $(f_1(x), \ldots, f_k(x)) \in S$ and consider the function $g(x) = q(f_1(x), \ldots, f_k(x)), x \in X$. Then $g \in \mathcal{N}(X)$ and $g$ is algebraic over $\mathbb{R}[f_1, \ldots, f_k]$. In particular, the ring $\mathcal{N}(X)$ of Nash functions on $X$ is algebraic over the ring $\mathbb{R}[X]$ of polynomials on $X$ and $\text{trdeg} \mathcal{N}(X) = \text{trdeg} \mathbb{R}[X] = \dim(X)$.

The proof of Proposition 2.8 is presented in Appendix A.

### 3. Nash systems

Nash systems are dynamical systems with inputs and outputs such that their state-spaces are Nash submanifolds and their dynamics and output functions are specified by Nash functions. We consider Nash systems with the following fixed input- and output-spaces. The input-space $U$ is a subset of $\mathbb{R}^m$. The output-space is $\mathbb{R}^r$. As the space of input functions for a Nash system with a given input-space $U$ we consider the set $U_{pc}$ of piecewise-constant functions $u : [0, T_u] \to U$, where $T_u \in [0, +\infty)$ is the maximal finite time instance for which $u$ is defined.

Any $u \in U_{pc}$ can be identified with a finite sequence $u = (\alpha_1, t_1) \cdots (\alpha_n, t_n)$ where $\alpha_i \in U, t_i \in [0, \infty)$ for $i = 1, \ldots, n$. Then, for $t \in [\sum_{j=0}^i t_j, \sum_{j=0}^{i+1} t_j)$, $u(t) = \alpha_{i+1} \in U$ for $i = 0, 1, \ldots, n - 1$, $t_0 = 0$, and $u(0) = \alpha_0$. Note that the same $u$ can have several different representations as a sequence of tuples $(\alpha, t)$ where $\alpha \in U, t \in [0, \infty)$. Every $u = (\alpha_1, t_1) \cdots (\alpha_n, t_n) \in U_{pc}$ has a time-domain $[0, T_u]$ where $T_u = \sum_{j=1}^n t_j$. If $u = (\alpha_1, t_1) \cdots (\alpha_n, t_n), v = (\beta_1, s_1) \cdots (\beta_k, s_k) \in U_{pc}$ then by $(u)v \in U_{pc}$ we denote the input $(\alpha_1, t_1) \cdots (\alpha_n, t_n)(\beta_1, s_1) \cdots (\beta_k, s_k)$. The empty input $e$ is the input with $T_e = 0$. Notice that $u(0) = v$ and $uv$ represent the same piecewise-constant input for any $u, v \in U_{pc}, \alpha \in U$. To express that the input $u$ was applied The empty input $e$ is the input with $T_e = 0$. only on the time-domain $[0, t] \subseteq [0, T_u]$ we write a subindex $(0, t]$ to $u$ like $u_{[0,t]}$.

**Definition 3.1.** A Nash system $\Sigma$ with an input-space $U \subseteq \mathbb{R}^m$ and an output-space $\mathbb{R}^r$ is a quadruple $(X, f, h, x_0)$ where

(i) the state-space $X$ is a Nash submanifold of $\mathbb{R}^n$ which is semi-algebraically connected,

(ii) the dynamics of the system is given by $\dot{x}(t) = f(x(t), u(t))$ for an input $u \in U_{pc}$, where $f : X \times U \to \mathbb{R}^n$ is such that for every input value $\alpha \in U$ the function $f_\alpha : X \ni x \mapsto f(x, \alpha) \in \mathbb{R}^n$ is a Nash function.

(iii) the output of the system is specified by a Nash function $h : X \to \mathbb{R}^r$,

(iv) $x_0 = x(0) \in X$ is the initial state of $\Sigma$. 

Note that Nash system $\Sigma$ can be considered a triple $\Sigma = (X, f, h)$ once one does not specify the initial state. Such definition is sufficient e.g. for the definition of observability properties of Nash systems, see Section 4.3.2. However, we keep Definition 3.1 because of the need to choose an initial state to solve the realization problem.

**Example 3.2.** The following simplified model of glycolysis and lactate production in Lactococcus lactis is adopted from the supplements of [40]:

$$
\begin{align*}
\dot{x}_1 &= 0.3592x_1^{-1}2.906_4^{0.2168}Glc^{1.1287} - 0.3115x_1^{2.17}ATP^{0.8152}, \\
\dot{x}_2 &= 0.3115x_1^{2.17}ATP^{0.8152} - 0.4698x_2^{1.0297}P_i^{0.2377}, \\
\dot{x}_3 &= 0.9396x_2^{1.0297}P_i^{0.2377} + 1.1452x_4^{3.5453} - 2.167x_3^{2.1649}, \\
\dot{x}_4 &= 2.167x_3^{2.1649} - 0.3592x_1^{-1}2.906_4^{0.2168}Glc^{1.1287} - 1.1452x_4^{3.5453} - 0.2087x_4^{0.0002} \\
&\quad - 0.9375x_2^{0.8744}x_4^{0.0091}P_i^{0.0005}, \\
\dot{x}_5 &= 0.3592x_1^{-1}2.906_4^{0.2168}Glc^{1.1287} - 0.0417x_5^{0.6202}x_2^{0.9264} - 1.3258x_5^{1.5255} \\
&\quad + 0.9375x_2^{0.8744}x_4^{0.0091}P_i^{0.0005}, \\
\dot{x}_6 &= 0.0417x_3^{0.6202}x_2^{0.9264}.
\end{align*}
$$

Here $x_1, \ldots, x_6$ are the concentrations of the respective metabolites (G6P, FBP, PGA3, PEP, pyruvate, lactate). The state-space $X = (0, +\infty)^6$ is a Nash submanifold of $\mathbb{R}^6$ which is semi-algebraically connected. The initial state can be chosen as $x_1(0) = \cdots = x_6(0) = 1$. The output function is the outflow of the pathway, i.e. $h(x_1, \ldots, x_6) = x_6$. The inputs are the concentrations of Glc, ATP and $P_i$ of external glucose, ATP, and inorganic phosphate, respectively.

Note that this system is neither polynomial nor rational, but still Nash. Indeed, $h$ is linear and hence Nash function of the state. The right-hand sides of the differential equations are linear combinations of terms of the form $x_1^{q_1} \cdots x_6^{q_n}$ where $q_i = 1, \ldots, 6$ is a rational number. Since Nash functions are closed under linear combination, multiplication and division, it remains to be shown that the map $g(x) = x^3$, where $x \in (0, +\infty)$ and $n, d \in \mathbb{N}$, is Nash. Since $g(x)$ is analytic, it is sufficient to show that $g$ is semi-algebraic. For all $x \in (0, +\infty)$ and for all $n, d \in \mathbb{N}$ it holds that $y = g(x)$ if and only if $P(y, x) = 0$, where $P$ is the polynomial defined as $P(Y, X) = Y^d - X^n$. Therefore, $g$ is a semi-algebraic function.

Note that in the example above, we did not allow zero concentrations. From a modeling point of view, this is not ideal, as in many biochemical applications zero concentrations may occur. The reason for this is that our definition of Nash systems requires the state-spaces to be Nash submanifolds without boundaries. If we allowed a state-space of the form $X = [0, \infty)^6$, then this state-space would be a Nash manifold with boundary. This is an inherent limitation of the class of Nash systems. At this stage, it is not clear if our results can be extended to systems, state-spaces of which are manifolds with boundaries. Perhaps one needs to follow a different approach to deal with such cases.

**Definition 3.3.** The state trajectory of a Nash system $\Sigma = (X, f, h, x_0)$ corresponding to an input $U_{pc} \ni u = (\alpha_1, t_1) \cdots (\alpha_k, t_k) : [0, T_u] \to U$ is a continuous piecewise-differentiable function $x_\Sigma(\cdot; x_0, u) : [0, T_u] \to X$ such that $x_\Sigma(0; x_0, u) = x_0$ and

$$
\frac{d}{dt} x_\Sigma(t; x_0, u) = f(x_\Sigma(t; x_0, u), u(t)) \quad (3.1)
$$

for $t \in (\sum_{j=0}^i t_j, \sum_{j=0}^{i+1} t_j)$, $i = 0, \ldots, k - 1$, $t_0 = 0$. 

\[ \tag{3.1} \]

\[ \text{for } t \in (\sum_{j=0}^i t_j, \sum_{j=0}^{i+1} t_j), \quad i = 0, \ldots, k - 1, \quad t_0 = 0. \]
Let $\Sigma = (X, f, h, x_0)$ be a Nash system with an input-space $U$. For any $\alpha \in U$, there exists a maximal $T > 0$ such that there exists a unique trajectory of $\Sigma$ corresponding to the constant input $u = (\alpha, T)$. This follows from the fact that $f(x, \alpha)$ is smooth with respect to $x$ for every $\alpha \in U$. Nevertheless, a unique solution of (3.1) need not exist for all $u \in U_{pc}$. In order to deal with this phenomenon we introduce the notion of admissible inputs for a Nash system.

**Definition 3.4.** A set $U_{pc}(\Sigma)$ of admissible inputs for a Nash system $\Sigma = (X, f, h, x_0)$ is a subset of $U_{pc}$ such that for all $u \in U_{pc}(\Sigma)$ there exists a unique trajectory $x_\Sigma(\cdot; x_0, u) : [0, T_u] \to X$ of $\Sigma$ corresponding to the input $u$.

4. Realization problem for Nash systems. In this section we formulate the realization problem for the class of Nash systems. First, in Section 4.1, we recall the notion of a response map adopted from [28, 29] and define when a Nash system is a realization of a response map. Next, in Section 4.2 and Section 4.3 we define minimality and semi-algebraic reachability and observability for Nash systems. These notions are necessary for formulating the realization problem for Nash systems. Finally, in Section 4.4 we state the realization problem for Nash systems.

4.1. Response maps. In order to formulate the realization problem, we need a formal definition of the input-output (external) behavior of the system. For our purposes, the response maps introduced in [28, 29] represent the right formalization of the external behavior of Nash systems. Below we recall their definition and basic properties.

**Definition 4.1.** A subset $\overline{U}_{pc}$ of the set $U_{pc}$ is called a set of admissible inputs if:

(i) $\forall u \in \overline{U}_{pc}$ $\forall t \in [0, T_u]$ : $u(0, t) \in \overline{U}_{pc}$,
(ii) $\forall u \in \overline{U}_{pc}$ $\forall \alpha \in U$ $\exists \delta > 0$ : $(u)(\alpha, t) \in \overline{U}_{pc}$,
(iii) $\forall u = (\alpha_1, t_1) \cdots (\alpha_k, t_k) \in \overline{U}_{pc}$ $\exists \delta > 0$ $\forall t_0 \in [0, t_1 + \delta]$ $i = 1, \ldots, k$ : $\overline{u} = (\alpha_1, t_1) \cdots (\alpha_k, t_k) \in U_{pc}$.

The first two conditions of the definition above assure the well-definedness of the derivative at switching times of differentiable functions which are defined on a set of admissible inputs, see Definition 5.1. The third condition allows us to prove Proposition 5.2 which is crucial for the proof of existence of Nash realizations.

In the sequel, unless stated otherwise, $U_{pc}$ denotes a set of admissible inputs.

**Definition 4.2.** We say that a function $\varphi : \overline{U}_{pc} \to R^r$ is a response map if for every sequence of input values $\alpha_1, \ldots, \alpha_k \in U$, $k > 0$, the function $\varphi_{\alpha_1, \ldots, \alpha_k}$, to be defined below, is analytic.

Let us denote by $T_{\alpha_1, \ldots, \alpha_k}$ the set of all $k$-tuples $(t_1, \ldots, t_k) \in [0, +\infty)^k$ such that the input $(\alpha_1, t_1)(\alpha_2, t_2) \cdots (\alpha_k, t_k)$ belongs to $U_{pc}$. We define the map $\varphi_{\alpha_1, \ldots, \alpha_k} : T_{\alpha_1, \ldots, \alpha_k} \to R^r$ as follows: for all $(t_1, \ldots, t_k) \in T_{\alpha_1, \ldots, \alpha_k}$

$$\varphi_{\alpha_1, \ldots, \alpha_k}(t_1, \ldots, t_k) = \varphi((\alpha_1, t_1) \cdots (\alpha_k, t_k)).$$

We denote the set of all response maps $\varphi : \overline{U}_{pc} \to R^r$ by $A(\overline{U}_{pc} \to R^r)$.

Let us point out the main difference between response maps and input-output maps, both describing external behavior of systems. Let $u : [0, T_u] \to R^n$ be an input to a system $\Sigma$ and let $y : [0, T_u] \to R^r$ be the corresponding output. Then a response map of $\Sigma$ maps $u$ to $y(T_u)$ while an input-output map of $\Sigma$ maps $u$ to $y$.

Consider a response map $p : \overline{U}_{pc} \to R^r$ and denote its coordinate functions by $p_i : \overline{U}_{pc} \to R$, $i = 1, \ldots, r$, i.e. $\forall u \in \overline{U}_{pc} : p(u) = (p_1(u), \ldots, p_r(u))$. Then
\( p_i \in \mathcal{A}(\tilde{U}_{pc} \rightarrow \mathbb{R}) \) for all \( i = 1, \ldots, r \).

We conclude this section by defining when a Nash system describes (i.e. realizes) a certain response map.

**Definition 4.3.** Consider a response map \( p : \tilde{U}_{pc} \rightarrow \mathbb{R}^r \). A Nash system \( \Sigma = (X, f, h, x_0) \) such that

\[ \tilde{U}_{pc} \subseteq U_{pc}(\Sigma) \quad \text{and} \quad p(u) = h(x_{\Sigma}(T_u; x_0, u)) \quad \text{for all} \quad u \in \tilde{U}_{pc} \]

is called a Nash realization of \( p \).

That is, \( \Sigma \) is a realization of \( p \), if

(a) all the inputs for which \( p \) is defined are admissible for \( \Sigma \), i.e. there exists a unique state trajectory corresponding to those inputs, and

(b) the value of \( p \) for the input \( u \) equals the output of \( \Sigma \) for \( u \) at time \( T_u \).

### 4.2. Minimal Nash realizations.

To state the realization problem for Nash systems formally we define the notion of dimension of a Nash system and the notion of minimal Nash realizations of response maps. These definitions are presented below.

**Definition 4.4.** By the dimension of a Nash system \( \Sigma = (X, f, h, x_0) \) we mean the dimension of its state-space \( X \), where \( X \) is viewed as a semi-algebraically connected Nash submanifold of \( \mathbb{R}^n \). We denote the dimension of \( \Sigma \) by \( \dim(\Sigma) \) and thus \( \dim(\Sigma) = \dim(X) \).

The dimension of \( \Sigma \) is related to the number of state variables as follows. If \( \dim(\Sigma) = d \) then there exists a local semi-algebraic coordinate transformation around any \( x \in X \) into \( \mathbb{R}^d \), i.e. locally, \( \Sigma \) can be described by \( d \) state variables. Notice that our definition of dimension is similar to the classical definition for nonlinear systems [16, 35].

**Definition 4.5.** We say that a Nash realization \( \Sigma = (X, f, h, x_0) \) of a response map \( p \) is a minimal Nash realization of \( p \) if for any Nash realization \( \Sigma' \) of \( p \) it holds that \( \dim(\Sigma) \leq \dim(\Sigma') \).

Note that if a Nash system is minimal according to the definition above, then this does not automatically imply that the number of states of that system is the smallest one among all the possible Nash realizations of the same input-output map. The reason for this is that we define the dimension of a Nash system as the dimension of its state-space. That is, if \( \Sigma \) is a \( d \)-dimensional Nash system, then its state-space is a \( d \)-dimensional Nash submanifold of the \( n \)-dimensional space \( \mathbb{R}^n \). Hence, its description will use \( n \) variables and \( n \) can be larger than \( d \). As it was pointed out before, if the dimension of the system is \( d \), then there exists a local coordinate transformation such that locally the system dynamics can be described by \( d \) variables. However, for the description of the global dynamics, one still needs \( n \) state variables. More precisely, in each coordinate chart (set \( \Omega_p, x \in X \) according to our notation), the system dynamics can be described by \( d \) coordinates, as long as the state stays in the coordinate chart. Once the state enters another coordinate chart, another description using \( d \) variables becomes valid. However, in order to describe the change of coordinate charts and thus describe the dynamics fully, one has to use all \( n \) original state variables.

This means that there might exist a minimal realization whose state-space is a \( d \)-dimensional submanifold of the \( n \)-dimensional space \( \mathbb{R}^n \), and a non-minimal realization whose state space is a \( d < d_1 \)-dimensional submanifold of the \( n_1 \)-dimensional space \( \mathbb{R}^{n_1} \), such that \( n > n_1 \). In this case, the **global** description of the dynamics of the minimal system will use \( n \) states, while the **global** description of the non-minimal system will use \( n_1 < n \) states. However, the **local** description of the minimal system will use less variables than the local description of the non-minimal system. Below we present an example to illustrate this point.
Example 4.6. Consider the Nash system \( \Sigma = (X, f, x_0) \) where \( U = \mathbb{R}, X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1, x_1 = x_2 \}, h(x_1, x_2, x_3) = \sqrt{1 + x_3^2(x_1^2 + x_2^2)} \), \( f((x_1, x_2, x_3), u) = (\sqrt{0.5u_3x_3}, \sqrt{0.5u_3x_3}, -\sqrt{2ux_1})^T \) and \( x_0 = (0, 0, 1) \).

It is easy to see that \( X \) is a Nash manifold and \( f_u \) and \( h \) are Nash functions and \( \dim(X) = 1 \). Indeed, define the sets \( \Omega_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0, x_1 \in (-\sqrt{0.5}, \sqrt{0.5}) \}, \Omega_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 < 0, x_1 \in (-\sqrt{0.5}, \sqrt{0.5}) \}, \Omega_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 > 0, x_3 \in (-1, 1) \}, \Omega_4 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 < 0, x_3 \in (-1, 1) \} \) and let \( \Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \in (-\sqrt{0.5}, \sqrt{0.5}) \} \). Define \( \psi_i : \Omega \to \Omega_i, \) \( i = 1, \ldots, 4 \) as follows:

\[
\psi_i(x_1, x_2, x_3) = \begin{cases} 
(x_1, x_1, \sqrt{1 - 2x_1^2}) & \text{if } i = 1 \\
(x_1, x_1, -\sqrt{1 - 2x_1^2}) & \text{if } i = 2 \\
(\sqrt{0.5(1 - 2x_1^2)}, \sqrt{0.5(1 - 2x_1^2)}, \sqrt{2x_1}) & \text{if } i = 3 \\
(-\sqrt{0.5(1 - 2x_1^2)}, -\sqrt{0.5(1 - 2x_1^2)}, \sqrt{2x_1}) & \text{if } i = 4
\end{cases}
\]

For each \( p = (p_1, p_2, p_3) \in X \), define the sets \( \Omega_p \), \( \Omega_p \subseteq \mathbb{R}^2 \) and the map \( \varphi_p : \Omega_p \to \Omega_p' \):

\[
\Omega_p = \begin{cases} 
\Omega_1 & \text{if } p \in \Omega_1 \\
\Omega_2 & \text{if } p \in \Omega_2 \\
\Omega_3 & \text{if } p = (\sqrt{0.5}, \sqrt{0.5}, 0) \\
\Omega_4 & \text{if } p = (-\sqrt{0.5}, -\sqrt{0.5}, 0)
\end{cases}
\]

\[
\varphi_p(x_1, x_2, x_3) = \begin{cases} 
\psi_1(p - (x_1, x_2, x_3)) & \text{if } p \in \Omega_1, i = 1, 2 \\
\psi_3(p_1, p_1, p_2 - (x_1, x_2, x_3)) & \text{if } p = (\sqrt{0.5}, \sqrt{0.5}, 0) \\
\psi_4(p_3, p_1, p_2 - (x_1, x_2, x_3)) & \text{if } p = (-\sqrt{0.5}, -\sqrt{0.5}, 0)
\end{cases}
\]

It is then easy to see that \( \varphi_p \) and \( \varphi_p^{-1} \) are Nash functions, and hence that \( \varphi_p \) is a Nash diffeomorphism. Moreover, \( \varphi_p(0, 0, 0) = p \) and \( \varphi_p(\mathbb{R} \times \{0\} \times \{0\}) \cap \Omega_p = X \cap \Omega_p \).

Hence, \( X \) is indeed a Nash manifold and it is of dimension \( d = 1 \). Because the maps \( \Omega_p \ni (x_1, x_2, x_3) \to f_u(\varphi_p(x_1, x_2, x_3)) \) and \( \Omega_p \ni (x_1, x_2, x_3) \to h(\varphi_p(x_1, x_2, x_3)) \) are obviously Nash, one concludes that \( f_u \) and \( h \) are Nash functions.

Notice that the response map of \( \Sigma \) can be described by

\[
p(u) = \sqrt{1 + \cos^2(\int_0^{T_u} u(s)ds) \sin^2(\int_0^{T_u} u(s)ds)},
\]

for all \( u \in \mathcal{U}_{pc} \), since

\[
x_\Sigma(T_u; x_0, u) = (\sqrt{0.5 \sin(\int_0^{T_u} u(s)ds)}, \sqrt{0.5 \sin(\int_0^{T_u} u(s)ds)}, \cos(\int_0^{T_u} u(s)ds))
\]

satisfies the differential equation of the system.

Clearly, \( \dim(\Sigma) = \dim(X) = d = 1 \) and hence \( \Sigma \) is a minimal realization. Notice that one uses three state variables to describe the system. However, if \( (x_1, x_2, x_3) \in (\Omega_1 \cup \Omega_2) \cap X \), then one variable \( (x_i) \) is sufficient to describe the dynamics of the system: \( h(x_1, x_2, x_3) = \sqrt{1 + x_3^2(1 - 2x_1^2)} \) and \( \dot{x}_1 = \sqrt{0.5u_1 \sqrt{1 - 2x_1^2}} \) for \( (x_1, x_2, x_3) \in \Omega_1 \) and \( \dot{x}_1 = -\sqrt{0.5u_1 \sqrt{1 - 2x_1^2}} \) for \( (x_1, x_2, x_3) \in \Omega_2 \). Similar local equations in one variable can be used if \( (x_1, x_2, x_3) \in (\Omega_i \cap X), i = 3, 4. \)
Consider now the system $\hat{\Sigma} = (\hat{X}, \hat{f}, \hat{h}, \hat{x}_0)$ where $\hat{X} = \mathbb{R}^2$, $\hat{x}_0 = (0,1)$, $\hat{h}(x_1, x_2) = \sqrt{1 + x_2^2 x_1^4}$, $\hat{f}(x_1, x_2, u) = (ux_2, -ux_1)$. It is easy to see that

$$x_{\hat{\Sigma}}(T_u; \hat{x}_0, u) = (\sin(\int_0^{T_u} u(s)ds), \cos(\int_0^{T_u} u(s)ds))$$

and hence $\hat{\Sigma}$ is a realization of $p$.

Note that $\dim(\hat{\Sigma}) = 2$ and hence $\hat{\Sigma}$ is not a minimal Nash realization of $p$. However, while the global description of $\Sigma$ involves 3 variables, $\hat{\Sigma}$ can be described by 2 variables.

Note that the phenomenon above is not entirely a matter of choosing the wrong definition of minimality. One expects the minimal system to be reachable and observable in some sense. We will also show that in fact $\hat{\Sigma}$ is not reachable, and hence it is not a good candidate for being a minimal realization.

Note that the situation described by Example 4.6 can also be encountered when considering analytic or smooth systems on manifolds, it is not specific to Nash systems.

4.3. Reachability and observability. In order to present the realization problem for Nash systems, we have to choose a suitable definition of reachability and observability for Nash systems. While there are standard definitions for both concepts, these definitions do not take into account the algebraic structure of the systems of interest. As a result, the precise definition of observability and reachability varies with the system class. For Nash systems, we will use semi-algebraic reachability and semi-algebraic observability instead of reachability and observability. We present these notions in Subsections 4.3.1 and 4.3.2. We chose these two concepts because they seemed to be useful for characterizing minimality of Nash systems and because they are closely related to the standard concepts of reachability and observability. It remains a topic of future research whether semi-algebraic reachability and observability are also useful for control design or state estimation.

4.3.1. Semi-algebraic reachability. We define semi-algebraic reachability of Nash realizations by a slight modification of the usual concept of reachability. Namely, instead of the requirement that the whole state-space is reachable from the initial state, we only require that the set of reachable states is, in a sense, a sufficiently large subset of the state-space.

**Definition 4.7.** Let $p : \tilde{U}_{pc} \rightarrow \mathbb{R}^r$ be a response map and let $\Sigma = (X, f, h, x_0)$ be a Nash realization of $p$. We denote by $\mathcal{R}(x_0)$ the set of states of $\Sigma$ reachable from $x_0$ by the inputs of $\tilde{U}_{pc}$, i.e.

$$\mathcal{R}(x_0) = \{x_{\Sigma}(T_u; x_0, u) \mid u \in \tilde{U}_{pc}\}.$$ 

Since $\tilde{U}_{pc} \subseteq U_{pc}(\Sigma)$, the reachable set $\mathcal{R}(x_0)$ is potentially smaller than the set of all states of $\Sigma$ reachable from $x_0$ by the inputs of $U_{pc}(\Sigma)$.

**Definition 4.8.** We say that a Nash realization $\Sigma = (X, f, h, x_0)$ of a response map $p : \tilde{U}_{pc} \rightarrow \mathbb{R}^r$ is semi-algebraically reachable, if no non-zero element of $\mathcal{N}(X)$ vanishes on the set of states of $\Sigma$ reachable from $x_0$, i.e. if

$$\forall g \in \mathcal{N}(X) : (g = 0 \text{ on } \mathcal{R}(x_0) \Rightarrow g = 0).$$
Example 4.9. The system $\Sigma$ from Example 4.6 is not semi-algebraically reachable: it is clear that for any $u \in U_{pc}$, $x_\Sigma(T_u; x_0, u)$ belongs to $\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$. Hence, the polynomial $P(x, y) = x^2 - y^2 - 1$ is zero on the set of states reachable from $x_0$. Since $P \neq 0$ and $P \in \mathcal{N}(\mathbb{R}^2)$, it follows that $\Sigma$ is not semi-algebraically reachable.

If a Nash system $\Sigma$ is reachable, i.e. if every state of $\Sigma$ can be reached by a suitable input, then $\Sigma$ is also semi-algebraically reachable provided that $U_{pc} = U_{pc}(\Sigma)$. Recall from [35, Section 4.3] that a nonlinear smooth system is called accessible if the set of reachable states contains an open set. Accessibility of nonlinear systems admits a characterization in terms of the rank of the Lie-algebra generated by the vector fields of the system. Since the definition of accessibility and the corresponding Lie-rank condition can directly be applied to Nash systems, the following proposition implies that Lie-rank condition yields a sufficient condition for semi-algebraic reachability.

Proposition 4.10. Let $\Sigma = (X, f, h, x_0)$ be a Nash realization of a response map $p : U_{pc} \to \mathbb{R}$. If $\Sigma$ is accessible, i.e. if there exists a non-empty open subset $S$ of $X$ such that $S \subseteq R(x_0)$, then $\Sigma$ is semi-algebraically reachable.

Proof. Let $g \in \mathcal{N}(X)$ be arbitrary. Because $S \subseteq R(x_0)$, $g = 0$ on $R(x_0)$ implies $g = 0$ on $S$. Since $S$ is a non-empty open subset of $X$ and since the Nash submanifold $X$ is semi-algebraically connected, from Proposition 2.6 it follows that $g = 0$. Thus, $\Sigma$ is semi-algebraically reachable.

We are now in position to show that the system $\Sigma$ from Example 4.6 is semi-algebraically reachable.

Example 4.11. It is easy to see that the state trajectory of $\Sigma$ from the initial state $x_0 = (0, 0, 1)$ equals

$$x_\Sigma(T_u; u, x_0) = (\sqrt{0.5} \sin \int_0^{T_u} u(s) ds, \sqrt{0.5} \sin \int_0^{T_u} u(s) ds, \cos \int_0^{T_u} u(s) ds).$$

Hence, states arising from constant inputs already yield the whole state-space $X$, i.e. $R(x_0) = X$ and in particular, $R(x_0)$ contains a subset which is open in the topology of $X$. Hence, by Proposition 4.10 $\Sigma$ is semi-algebraically reachable.

The converse of Proposition 4.10 is not true, as demonstrated by the following example. That is, it can happen that the system is semi-algebraically reachable, even though the set of reachable states has an empty interior.

Example 4.12. Consider the system $\Sigma = (X, f, h, x_0)$ where $U = \mathbb{R}$, $X = \mathbb{R}^2$, $x_0 = (0, 1)$, $\forall u \in U$, $f((x_1, x_2), u) = (1, x_2)$ and $h(x_1, x_2) = x_1 + x_2^2$. It is easy to see that $x_\Sigma(T_u; u, x_0) = (T_u, e^{T_u})$. In other words, $R(x_0) = \{ (t, e^t) \mid t \in [0, +\infty) \}$. Since $R(x_0)$ is a graph of a smooth function of time, it cannot contain an open subset of $\mathbb{R}^2$. We will show that $\Sigma$ is semi-algebraically reachable. To this end, consider a Nash function $0 \neq g \in \mathcal{N}(\mathbb{R}^2)$ and assume that $g|_{R(x_0)} = 0$, that is, $g(t, e^t) = 0$ for all $t \geq 0$. Since $g$ is an analytic function, the map $\mathbb{R} \ni t \mapsto g(t, e^t)$ is analytic and hence $g(t, e^t) = 0$ should hold for all $t \in \mathbb{R}$. Since $g$ is a semi-algebraic function, we can apply Lemma 2.1 to it. Then there exist finitely many polynomials $g_i$ and semi-algebraic sets $S_i$, $i = 1, \ldots, d$, such that $S_i$ are disjoint, $g_i(x, Y) \neq 0$ for all $x \in S_i$, $\bigcup_{i=1}^d S_i = \mathbb{R}^2$ and $g_i(x, g(x)) = 0$ for all $x \in S_i$. Define $Q(X_1, X_2, Y) = g_1(X_1, X_2, Y) \cdots g_d(X_1, X_2, Y)$. It is clear that $Q \neq 0$. It then follows that $Q(x, g(x)) = 0$ on $\mathbb{R}^2$. If $Q(X_1, X_2, 0) = 0$, then for some $k > 0$, $Q(X_1, X_2) = Y^k R(X_1, X_2, Y)$, where $R(X_1, X_2, Y)$ is a polynomial such that $R(X_1, X_2, 0) \neq 0$. It then follows that $(g(x))^{k} R(x, g(x)) = 0$ on $\mathbb{R}^2$. Since the ring of entire analytic functions is an integral domain, it follows that either $g = 0$ or $R(x, g(x)) = 0$ for all $x$. Since $g \neq 0$, it then follows that $R(x, g(x)) =$
for all \( f \) is closed under taking Lie-derivatives with respect to the vector fields \( f \).

In other words, the function \( e^t \) is an algebraic function. But the exponential function is known to be transcendental, hence this is a contradiction.

That is, there cannot exist a non-zero Nash function \( g \in \mathcal{N}(\mathbb{R}^2) \) such that \( g \) vanishes on \( \mathcal{R}(x_0) \).

### 4.3.2. Semi-algebraic observability

Below we define the concept of semi-algebraic observability for Nash systems. The proposed concept was inspired by the notion of algebraic observability of \([4, 28, 29]\). Since the to be defined concepts do not depend on the initial state, in this section, unless stated otherwise, a Nash system will mean a tuple \( \Sigma = (X, f, h) \) without the initial state. First, we will define the concept of observation algebra for Nash systems.

**Definition 4.13.** The observation algebra \( A_{\text{obs}}(\Sigma) \) of a Nash system \( \Sigma = (X, f, h) \) is the smallest subalgebra of \( \mathcal{N}(X) \) which contains \( h_i \), \( i = 1, \ldots, r \) and which is closed under taking Lie-derivatives with respect to the vector fields \( f_{\alpha} : X \ni x \mapsto f(x, \alpha) \in \mathbb{R}^n \), \( \alpha \in U \).

The definition of the observation algebra of a Nash system is analogous to the one of rational and polynomial systems presented in \([3, 4, 41]\).

**Definition 4.14.** We say that a Nash system \( \Sigma = (X, f, h) \) is semi-algebraically observable if \( \text{trdeg} \ A_{\text{obs}}(\Sigma) = \text{trdeg} \ N(X) \). We will say that a Nash system \( \Sigma = (X, f, h, x_0) \) equipped with an initial state is semi-algebraically observable, if \( (X, f, h) \) is semi-algebraically observable.

That is, \( \Sigma \) is semi-algebraically observable, if the algebra of all Nash functions on \( X \) is algebraic over the observation algebra of \( \Sigma \). This means that any Nash function is a solution of a polynomial equation, coefficients of which are elements of the observation algebra.

Another way to gain intuition for the concept of semi-algebraic observability is to use the analogy with linear system theory. For linear systems, the observation algebra corresponds to the algebra generated by the rows of the observability matrix, and the transcendence degree corresponds to the rank of the observability matrix. Hence, for linear systems, semi-algebraic observability boils down to the Kalman rank condition for observability. In other words, the notion of semi-algebraic observability can be viewed as a generalization of the Kalman rank condition for observability.

To illustrate the concept further, we will review the system of Example 4.6.

**Example 4.15.** Consider the system \( \Sigma = (X, f, h, x_0) \) from Example 4.6. It is easy to see that the Nash manifold \( X \) is connected and hence by Lemma 2.8, \( \text{trdeg} \ N(X) = \dim(X) = 1 \). Hence, if we can show that \( \text{trdeg} \ A_{\text{obs}}(\Sigma) = 1 \), then it follows that \( \text{trdeg} \ A_{\text{obs}}(\Sigma) = \text{trdeg} \ N(X) \). To this end, it is enough to show that there exists no polynomial in one variable \( 0 \neq Q \in \mathbb{R}[T] \), such that \( Q(h) = 0 \) on \( X \). Indeed, \( Q(h(x)) = 0 \) for all \( x \in X \) implies that \( h(x) \) is a root of \( Q \). But \( Q \) can have only finitely many roots and hence \( h(x) \), should take only finitely many values. However, this is clearly not the case.

In order to connect the concept of semi-algebraic observability to the standard definition of observability, we introduce the notion of strong semi-algebraic observability. If \( \Sigma \) is strongly semi-algebraically observable, then it is observable in the sense that there are no indistinguishable states, and it is also semi-algebraically observable. The details are as follows.

**Definition 4.16.** The Nash extension \( A_{\text{Nash}}^N(X) \) of the observation algebra \( A_{\text{obs}}(\Sigma) \) of a Nash system \( \Sigma = (X, f, h) \) is defined as the sub-algebra of \( N(X) \) gener-
ated by Nash functions $g : X \to \mathbb{R}$ of the form
$$\forall x \in X : g(x) = q(\varphi_1(x), \ldots, \varphi_k(x))$$
where
- $\varphi_1, \ldots, \varphi_k \in A_{\text{obs}}(\Sigma)$, $k > 0$
- $q \in \mathcal{N}(S)$ for some Nash manifold $S \subseteq \mathbb{R}^k$, such that
$$\forall x \in X : (\varphi_1(x), \ldots, \varphi_k(x)) \in S.$$

The concept of Nash extension formulated in Definition 4.16 is a slight generalization of a similar concept from [27].

**Definition 4.17.** We say that a Nash system $\Sigma = (X, f, h)$ is strongly semi-algebraically observable if $A_{\text{Nash}}(\Sigma) = \mathcal{N}^*(X)$. We say that a Nash system $(X, f, h, x_0)$ equipped with an initial state is strongly semi-algebraically observable, if $(X, f, h)$ is strongly semi-algebraically observable.

Intuitively, strong semi-algebraic observability means that any Nash function can be obtained by substituting some elements of the observation algebra into a Nash function which is defined on some semi-algebraic set. This is analogous to the definition of algebraic observability in [4, 28, 29]: algebraic observability means that any polynomial (rational) function defined on the state-space can be obtained by substituting the elements of the observation algebra into a polynomial (rational) function.

Strong semi-algebraic observability and semi-algebraic observability are related as follows.

**Proposition 4.18.** If a Nash system is strongly semi-algebraically observable then it is semi-algebraically observable.

Note that the converse of Proposition 4.18 is not true, as it is demonstrated by the following counter-example.

**Example 4.19.** Consider the system $\Sigma$ from Example 4.12. It then follows that $L_{f \alpha}(h) = 1 + 2x_2^2$ and $L_{f \beta_1} \cdots L_{f \beta_k} h = 2^k x_2^2$ for all $k > 1$. Hence, $A_{\text{obs}}(\Sigma) = \mathbb{R}[x_1, x_2^2]$. Since the polynomials $x_1$ and $x_2^3$ are trivially algebraically independent, it follows that $\text{trdeg} A_{\text{obs}}(\Sigma) = \text{trdeg} \mathcal{N}(\mathbb{R}^2) = \dim(\mathbb{R}^2) = 2$. Assume now that $\Sigma$ is strongly semi-algebraically observable. Then there exists a Nash manifold $S \subseteq \mathbb{R}^k$ and a Nash function $g \in \mathcal{N}(S)$, and $k$ polynomials $R_1, \ldots, R_k$, such that $x_2 = g(R_1(x_1, x_2^2), \ldots, R_k(x_1, x_2^2))$ for all $x_1, x_2 \in \mathbb{R}$. Since the composition of Nash functions with polynomials is again a Nash function, without loss of generality we can assume that $x_2 = g(x_1, x_2^2)$ and $k = 2$. Since $\frac{\partial}{\partial x_1} x_2 = \frac{\partial}{\partial x_1} g(x_1, z) = 0$, it then follows that we can take $k = 1$ and assume that $x_2 = g(x_2^2)$. But then $1 = g(1^2) = g((-1)^2) = -1$, a contradiction. Hence, clearly $x_2$ cannot belong to $A_{\text{Nash}}^*(\Sigma)$. This means that $\Sigma$ cannot be strongly semi-algebraically observable.

Notice that if in the counter-example above we take $X = \mathbb{R} \times (0, +\infty)$, then the thus obtained system is strongly semi-algebraically observable. In fact, we can show the following.

**Proposition 4.20.** If $\Sigma = (X, f, h)$ is semi-algebraically observable, then there exists an open semi-algebraic subset $V$ of $X$ such that the system $\Sigma_V = (V, f|_V, h|_V)$ is strongly semi-algebraically observable. We denote by $f^V|_V$ the restriction of $f_a$ to $V$, and by $h|_V$ the restriction of $h$ to $V$.

Recall from [8] that any open semi-algebraic subset of a Nash submanifold of dimension $d$ is itself a Nash submanifold of dimension $d$.

A system-theoretic interpretation of strong semi-algebraic observability is provided by the following proposition and its corollary.
PROPOSITION 4.21. If a Nash system \( \Sigma \) is strongly semi-algebraically observable, then any two states \( x_1 \neq x_2 \) of \( \Sigma \) are distinguishable by an element of \( A_{obs}(\Sigma) \), i.e. \( \exists g \in A_{obs}(\Sigma) : g(x_1) \neq g(x_2) \).

COROLLARY 4.22. Let \( \Sigma = (X, f, h) \) be a strongly semi-algebraically observable Nash system. Then \( \Sigma \) is observable in the sense that it has no indistinguishable states, i.e. if \( x_1 \neq x_2 \in X \) then there exists \( u \in U_{pc} \) such that \( x_\Sigma(T_u; x_1, u) \) and \( x_\Sigma(T_u; x_2, u) \) are both defined and \( h(x_\Sigma(T_u; x_1, u)) \neq h(x_\Sigma(T_u; x_2, u)) \).

Corollary 4.22 implies that differential-geometric conditions for observability of nonlinear systems, see for example [16], also yield necessary conditions for strong semi-algebraic observability of Nash systems. The proofs of Proposition 4.18, Proposition 4.20, Proposition 4.21 and Corollary 4.22 can be found in Appendix A.

Recall [16] that \( \Sigma = (X, f, h) \) is said to satisfy the observability rank condition, if the rank of the observability co-distribution \( \forall x \in X : \Delta(x) = \text{span}\{d(L_{f,u_1} \cdots L_{f,u_k} h(x) \mid \alpha_1, \ldots, \alpha_k \in U, k \geq 0)\} = \dim(X) \), i.e. \( \dim(\Delta(x)) = \dim(X) \) for all \( x \in X \).

Recall from [15, Theorem 3.1] that if \( \Sigma \) satisfies the observability rank condition, then it is locally weakly observable. By local weak observability we mean that for any state \( x \) there is an open neighborhood \( V \) of \( x \) in \( X \), such that for any open subset \( W \subseteq V \) and any \( x_1 \in W \), there exists an \( u \in U_{pc} \) such that \( x_\Sigma(; u, x) \) and \( \forall t \in [0, T_u] : \Delta(t; u, x) \) exist and \( h(x_\Sigma(T_u; u, x_1)) \neq h(x_\Sigma(T_u; u, x_2)) \). For analytic systems, and hence for Nash systems, local weak observability is known (\([7, \text{Theorem 2.1}]\)) to be equivalent to weak observability: \( \Sigma \) is called weakly observable, if for any \( x \in X \) there exists an open neighborhood \( V \) of \( x \) such that for any \( u \in U_{pc} \) : \( x_\Sigma(; u, x) \) exist, and \( h(x_\Sigma(T_u; u, x_1)) \neq h(x_\Sigma(T_u; u, x_2)) \). From [15, Theorem 3.1] it follows that if the system is locally weakly observable, then there exists an open subset \( W \) of \( X \) such that for all \( x \in W : \dim(\Delta(x)) = \dim(X) \).

PROPOSITION 4.23. Let \( \Sigma = (X, f, h) \) be a Nash system. Then \( \Sigma \) is semi-algebraically observable if and only if there exists \( x \in X \), \( \dim(\Delta(x)) = \dim(X) \). In particular, if \( \Sigma \) is locally weakly observable or if \( \Sigma \) satisfies the observability rank condition, then it is semi-algebraically observable.

The proof of Proposition 4.23 is presented in Appendix A. Note that while local weak observability (observability rank condition) implies semi-algebraic observability, the converse is not true in general, see the counter-example below.

EXAMPLE 4.24. Consider the system \( \Sigma \) from Example 4.19. It is easy to see that \( h(x_\Sigma(T_u; (0,0), u)) = h(x_\Sigma(T_u; (0,1), u)) = T_u + e^{2T_u}x^2 \). Hence, in any neighborhood \( V \) of \((0,0)\), any two elements of the form \((0,0), (0,x)\) are indistinguishable. Hence, \( \Sigma \) is not locally weakly observable. It follows that the observability co-distribution is \( \Delta(m) = \text{span}\{dx(m), dy(m)\} \) for any \( m \in \mathbb{R}^2 \). But \( \dim(\Delta(0)) = \dim(\text{span}\{dx(0)\}) = 1 \neq 2 \), hence \( \Sigma \) does not satisfy the observability rank condition. However, in Example 4.19 we have already shown that \( \Sigma \) is semi-algebraically observable.

4.4. Problem formulation. The realization problem for Nash systems consists of three subproblems which concern the existence and minimality of Nash realizations and the algorithms for computing them.

PROBLEM 4.25. Let \( p : U_{pc} \to \mathbb{R}^r \) be a response map. The Nash realization problem for \( p \) consists of the following subproblems:

Existence Determine necessary and sufficient conditions for the existence of a Nash realization of \( p \).

Minimality Determine necessary and sufficient conditions for the existence of a minimal Nash realization of \( p \). Determine whether a minimal Nash realization of \( p \) is
unique in any sense (for example, up to an isomorphism). Determine the relationship between minimality, observability and reachability.

**Realization algorithm** Formulate the algorithms for computing a (minimal) Nash realization of $p$ from finite data directly obtainable from $p$. In addition, formulate algorithms for checking minimality of a Nash realization and for transforming a Nash realization of $p$ to a minimal one.

In this paper we deal with the existence and minimality problems for Nash realizations. We do not address the algorithmic part of the realization problem.

5. **Main results.** In this section we present the main results of the paper. We will start with conditions for existence of a Nash realization and we will continue by addressing minimality of Nash systems.

5.1. **Existence of a Nash realization.** In order to state necessary and sufficient conditions for the existence of a Nash realization of a response map we need to define a number of concepts for response maps. First, we recall the notion of a derivative of a response maps from [4, 28, 29].

**Definition 5.1** ([4]). Consider a function $\varphi \in \mathcal{A}(\overline{U}_{pc} \to \mathbb{R})$. For any $\alpha \in U$ we define the derivative

$$
(D_{\alpha}\varphi)(u) = \frac{d}{dt}\varphi((u)(\alpha,t))_{|t=0^+} \text{ for all } u \in \overline{U}_{pc}.
$$

Next, we recall the fact that the set of all response maps forms an algebra, in fact it is an integral domain.

**Proposition 5.2** ([28, 29]). The set $\mathcal{A}(\overline{U}_{pc} \to \mathbb{R})$ with point-wise addition and multiplication forms a commutative algebra over $\mathbb{R}$ and it is an integral domain.

Further, we recall the notion of observation algebra of a response map.

**Definition 5.3** ([4]). Let $p : \overline{U}_{pc} \to \mathbb{R}$ be a response map. The observation algebra $A_{obs}(p)$ of $p$ is the smallest subalgebra of the algebra $\mathcal{A}(\overline{U}_{pc} \to \mathbb{R})$ which contains the components $p_i, i = 1, \ldots, r$ of $p$, and which is closed with respect to the derivations $D_{\alpha}, \alpha \in U$. Thus, $A_{obs}(p)$ is the smallest algebra such that $p_1, \ldots, p_r \in A_{obs}(p)$, and if $\varphi \in A_{obs}(p)$ then $D_{\alpha}\varphi \in A_{obs}(p)$ for all $\alpha \in U$.

Another key notion is the one of Nash extension of a finite subset of the set $\mathcal{A}(\overline{U}_{pc} \to \mathbb{R})$ of scalar-valued response maps. Recall that the concept of Nash extension of an infinite set of functions is introduced in Definition 4.16 for observation algebras. The following definition of Nash extension of a finite set of functions is then a modification of Definition 4.16.

**Definition 5.4.** Let $X$ be a Nash submanifold of $\mathbb{R}^n$ which is semi-algebraically connected, let $U_{pc}$ be a set of admissible inputs, and let $\mathcal{A} = \{\varphi_1, \ldots, \varphi_n\}$ be a subset of $\mathcal{A}(\overline{U}_{pc} \to \mathbb{R})$. Assume that for all $u \in \overline{U}_{pc}$, $(\varphi_1(u), \ldots, \varphi_n(u)) \in X$. The Nash extension $\mathcal{A}^{Nash}(X)$ of $\mathcal{A}$ with respect to $X$ is the subalgebra of $\mathcal{A}(\overline{U}_{pc} \to \mathbb{R})$ generated by the maps $\overline{U}_{pc} \ni u \mapsto q(\varphi_1(u), \ldots, \varphi_n(u)) \in \mathbb{R}$, where $q \in \mathcal{N}(X)$.

Intuitively, the Nash extension of $\mathcal{A}$ is obtained by substituting the elements of $\mathcal{A}$ into Nash functions defined on $X$. Note that the set of substitutions into linear forms (polynomials) yields the linear space (algebra) generated by $\mathcal{A}$. Then, the Nash extension of $\mathcal{A}$ can be thought of as the generalization of the notions of linear space and algebra generated by $\mathcal{A}$.

We are now able to state the following necessary and sufficient condition for the existence of a Nash realization.
THEOREM 5.5. A response map $p : \mathcal{U}_{pc} \to \mathbb{R}^r$ has a Nash realization if and only if there exist a finite subset $A = \{\varphi_1, \ldots, \varphi_n\}$ of $\mathcal{A}(\mathcal{U}_{pc} \to \mathbb{R})$ and a Nash submanifold $X \subseteq \mathbb{R}^n$ which is semi-algebraically connected such that

(i) $\forall u \in \mathcal{U}_{pc} : (\varphi_1(u), \ldots, \varphi_n(u)) \in X$,
(ii) $p_i \in \mathcal{A}^{N\text{nash}}(X)$ for $i = 1, \ldots, r$,
(iii) $\forall \varphi \in \mathcal{A}^{N\text{nash}}(X) \forall u \in U : D_\alpha \varphi \in \mathcal{A}^{N\text{nash}}(X)$.

The proof of Theorem 5.5 is presented in Section 6.

REMARK 5.6. The idea of the proof is very similar to that of [6, Theorem 4.6], where an analogous result for analytic systems on time scales is stated. Since the system class of [6] is much more general, the result of [6, Theorem 4.6] does not imply Theorem 5.5. More precisely, when applied to continuous-time systems, the results of [6, Theorem 4.6] provide necessary and sufficient conditions for the existence of an analytic realization. The corresponding conditions are the same as in Theorem 5.5, except for the usage of the analytic closure $\mathcal{A}^{N\text{nash}}(X)$ in place of $\mathcal{A}^{N\text{nash}}(X)$ and $\mathcal{A}^{N\text{nash}}(X)$.

Example 4.6. Let $\mathbb{R}^2$ be arbitrary. Then

\[
D_\alpha \varphi_1(u) = \frac{d}{dt} \left[ \sin(\int_0^{T_u} u(t)dt) \cos(\int_0^{T_u} u(t)dt) \right]_{t=0^+} = \alpha \cos(\int_0^{T_u} u(t)dt) = \alpha \varphi_2(u)
\]

and similarly

\[
D_\alpha \varphi_2(u) = -\alpha \sin(\int_0^{T_u} u(t)dt) = -\alpha \varphi_1(u).
\]

Let $\varphi \in \mathcal{A}^{N\text{nash}}(X)$ be arbitrary. Then $\varphi(u) = q(\varphi_1(u), \varphi_2(u))$, where $q \in \mathcal{N}(X)$ and $D_\alpha \varphi(u) = \frac{\partial q}{\partial \varphi_1}(\varphi_1(u), \varphi_2(u)) D_\alpha \varphi_1(u) + \frac{\partial q}{\partial \varphi_2}(\varphi_1(u), \varphi_2(u)) D_\alpha \varphi_2(u) = \alpha \frac{\partial q}{\partial \varphi_1}(\varphi_1(u), \varphi_2(u)) \varphi_2(u) - \alpha \frac{\partial q}{\partial \varphi_2}(\varphi_1(u), \varphi_2(u)) \varphi_1(u)$, where $\frac{\partial q}{\partial \varphi_1}, \frac{\partial q}{\partial \varphi_2} \in \mathcal{N}(X)$ and thus $D_\alpha \varphi \in \mathcal{A}^{N\text{nash}}(X)$. 

Example 5.7. Consider the response map

\[
p(u) = \sqrt{1 + \sin^2(\int_0^{T_u} u(t)dt) \cos^2(\int_0^{T_u} u(t)dt)}.
\]

where $u \in \mathcal{U}_{pc}$ are real-valued piecewise-constant inputs. Let us denote $\varphi_1(u) = \sin(\int_0^{T_u} u(t)dt)$ and $\varphi_2(u) = \cos(\int_0^{T_u} u(t)dt)$. Then, for all $u \in \mathcal{U}_{pc}$

\[
(\varphi_1(u), \varphi_2(u)) \in \{(x, y) : x^2 + y^2 = 1\} = X.
\]

Note that $X$ is semi-algebraically connected Nash submanifold of $\mathbb{R}^2$, see for example Example 4.6.

Since $p(u) = \sqrt{1 + \varphi_1^2(u)\varphi_2^2(u)}$, it follows that $p \in \mathcal{A}^{N\text{nash}}(X)$. Let $u \in \mathcal{U}_p$ be arbitrary. Then

\[
D_\alpha \varphi_1(u) = \frac{d}{dt} \left[ \sin(\int_0^{T_u} u(t)dt) + \int_{T_u}^{T_u+\tau} u(t)dt \right]_{\tau=0^+} = \alpha \cos(\int_0^{T_u} u(t)dt) = \alpha \varphi_2(u)
\]

and similarly

\[
D_\alpha \varphi_2(u) = -\alpha \sin(\int_0^{T_u} u(t)dt) = -\alpha \varphi_1(u).
\]
Therefore, the map \( p \) satisfies the conditions of Theorem 5.5 and hence its Nash realization does exist. We construct the realization in the same way as it is done in the proof of Theorem 5.5. Namely, \( X = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \} \) is considered the state space of the realization \( \Sigma \). The dynamics of \( \Sigma \) is defined according to the derivatives \( D_\alpha \varphi_1(u) = \alpha \varphi_2(u) \) and \( D_\alpha \varphi_1(u) = -\alpha \varphi_1(u) \) as follows:

\[
\dot{x} = uy, \quad \dot{y} = -ux.
\]

The output function \( h(x, y) = \sqrt{1 + x^2y^2} \) is based on the formula defining the dependence of \( p \) on \( \varphi_1 \) and \( \varphi_2 \). Finally, the initial states are \( x(0) = 0, \ y(0) = 1 \) which follows from \( \varphi_1(e) = 0, \ \varphi_2(e) = 1 \), respectively.

The conditions of Theorem 5.5 are rather difficult to check. Therefore, we will derive a simpler necessary condition for the existence of a Nash realization. In order to state this condition, recall that \( \mathcal{A}(\mathcal{U}_{pc} \to \mathbb{R}^d) \) is an integral domain, and hence so is the observation algebra of \( p \). Therefore, the transcendence degree \( \text{trdeg} A_{obs}(p) \) of the observation algebra of \( p \) is well-defined.

**Theorem 5.8.** Let \( \Sigma = (X, f, h, x_0) \) be a Nash realization of a response map \( p : \mathcal{U}_{pc} \to \mathbb{R}^d \). Then, \( \text{trdeg} A_{obs}(p) \leq \dim(X) \). Hence, if there exists a Nash realization of a response map \( p \) then \( \text{trdeg} A_{obs}(p) < +\infty \).

The proof of Theorem 5.8 is presented in Section 6.

This necessary condition is analogous to the finite Hankel-rank condition for linear systems. In particular, the transcendence degree of \( A_{obs}(p) \) can be seen as a generalization of the rank of the Hankel-matrix for linear systems.

**Example 5.9.** Consider the map \( p(u) = \sqrt{1 + \sin^2(\int_0^1 u(t)dt) + \cos^2(\int_0^1 u(t)dt)} \) which is realizable by a Nash system, see Example 5.7. We show that \( \text{trdeg} A_{obs}(p) < +\infty \). More precisely, we show that \( \text{trdeg} A_{obs}(p) = 1 = \dim(X) \), where \( X \) stands for the state space of the realization computed in Example 5.7.

First, let us compute \( A_{obs}(p) \). Recall that \( p = \sqrt{1 + \varphi_1^2 \varphi_2^2} \) and \( \varphi_1^2 + \varphi_2^2 = 1 \). Because \( p \in A_{obs}(p) \), one gets \( \varphi_1^2 \varphi_2^2 \in A_{obs}(p) \). Because \( D_\alpha (\varphi_1^2 \varphi_2^2) = 2\alpha \varphi_1 \varphi_2 \varphi_3 - 2\alpha \varphi_1^3 \varphi_2 \), one obtains \( \varphi_1 \varphi_3 - \varphi_1 \varphi_2 \in A_{obs}(p) \). Taking derivatives again, it is easy to see that \( D_\alpha (\varphi_1 \varphi_3 - \varphi_1 \varphi_2) = \alpha (\varphi_1^2 + \varphi_2^2) - 6\alpha \varphi_1^2 \varphi_2^2 \). Thus, \( \varphi_1^2 + \varphi_2^2 \in A_{obs}(p) \). Further, \( D_\alpha (\varphi_1^2 + \varphi_2^2) = -4\alpha (\varphi_1^3 \varphi_2^3 - \varphi_1 \varphi_2^4) \in A_{obs}(p) \). Consider now the algebra \( R = \mathbb{R}(\varphi_1^2 \varphi_2, \varphi_1^2, \varphi_2^2, \varphi_1 \varphi_3, \varphi_1 \varphi_2, \varphi_1^2 + \varphi_2^2) \). It then follows that \( R \) is closed under derivations \( D_\alpha \), i.e. if \( r \in R \), then \( D_\alpha (r) \in R \). It is easy to see that \( D_{\alpha_1} \cdots D_{\alpha_n} p \) are algebraic combinations of derivations of \( \varphi_1^2 \varphi_2^2 \) and (positive or negative) powers of \( \sqrt{1 + \varphi_1^2 \varphi_2^2} \). Hence, \( R \subseteq A_{obs}(p) \subseteq R[\sqrt{1 + \varphi_1^2 \varphi_2^2}, \frac{1}{\sqrt{1 + \varphi_1^2 \varphi_2^2}}] \). Since both \( R \) and \( R[\sqrt{1 + \varphi_1^2 \varphi_2^2}, \frac{1}{\sqrt{1 + \varphi_1^2 \varphi_2^2}}] \) are finitely generated, it then follows that their transcendence degrees are finite. Hence, the transcendence degree of \( A_{obs}(p) \) is finite too. It is easy to see that \( \sqrt{1 + \varphi_1^2 \varphi_2^2}, \frac{1}{\sqrt{1 + \varphi_1^2 \varphi_2^2}} \) are algebraic over \( R \): for \( p(x, y) = y - x^2 - 1 \),

\[
p(\sqrt{1 + \varphi_1^2 \varphi_2^2}, \varphi_1^2 \varphi_2, \varphi_1 \varphi_3, \varphi_1 \varphi_2, \varphi_1 \varphi_3, \varphi_1 \varphi_2) = 0.
\]

Similarly, for \( p(x, y) = x^2(y+1)-1, p(\frac{1}{\sqrt{1 + \varphi_1^2 \varphi_2^2}}, \varphi_1^2 \varphi_2) = 0 \).

Hence, \( \text{trdeg} A_{obs}(p) \leq \text{trdeg} R[\sqrt{1 + \varphi_1^2 \varphi_2^2}, \frac{1}{\sqrt{1 + \varphi_1^2 \varphi_2^2}}] = \text{trdeg} R \). Finally, notice that \( \varphi_1^2 + \varphi_2^2 = (\varphi_1^2 + \varphi_2^2)^2 - 2\varphi_1 \varphi_2^2 = 1 - \varphi_1^2 \varphi_2^2 \) and that \( s = \varphi_1 \varphi_2^3 - \varphi_1^3 \varphi_2 = \varphi_1 \varphi_2 (\varphi_1^2 - \varphi_2^2) \) satisfies \( p(s, \varphi_1^2 \varphi_2) = 0 \) for \( p(x, y) = x^2 - y^2(1 - 4y) \). Hence, \( R \) is algebraic over \( R[\sqrt{1 + \varphi_1^2 \varphi_2^2}] \) and hence \( \text{trdeg} R = \text{trdeg} A_{obs}(p) = 1 \).

We conjecture that the finiteness of the transcendence degree of \( A_{obs}(p) \) is also a sufficient condition for the existence of a Nash realization, at least provided that \( p \)
is defined for inputs whose values vary only within a finite input-space $U$. However, this statement is still an open problem.

What we can prove is the following. We will say that a response map $p$ is \textit{locally realized} by a Nash system $\Sigma = (X, f, h, x_0)$ if for any integer $k > 0$, $\alpha_1, \ldots, \alpha_k \in U$, there exists $t_i > 0$, $i = 1, \ldots, k$ such that for all $t_i \in [0, t_i]$, $i = 1, \ldots, k$, the input $v = (\alpha_1, t_1) \cdots (\alpha_k, t_k)$ satisfies $v \in U_{\text{pc}}(\Sigma)$, and

$$p(v) = h(x_{\Sigma}(T_v; x_0, v)).$$

That is, $\Sigma$ is a local realization of $p$, if for any piecewise-constant input with small enough switching times, the response of $p$ to this input equals the response of $\Sigma$ to this input. Further, if $u \in U_{\text{pc}}$, then denote by $p_u$ the map $p_u : U_{\text{pc}} \rightarrow \mathbb{R}'$, where $U_{\text{pc}} = \{v \in U_{\text{pc}} \mid uv \in U_{\text{pc}}\}$, and

$$\forall v \in U_{\text{pc}}^{-u} : p_u(v) = p(uv).$$

It is easy to see that $p_u$ is again a response map.

\textbf{Theorem 5.10.} Assume $\text{trdeg} \ A_{\text{obs}}(p) = d < +\infty$ and $U$ is finite. For any $\epsilon > 0$ one can choose an input $u \in U_{\text{pc}}$ and a Nash system $\Sigma$ such that $T_u < \epsilon$ and $\Sigma$ is a local realization of $p_u$.

The proof of Theorem 5.10 is presented in Section 6. Theorem 5.10 implies that if the observation algebra has a finite transcendence degree, then we can find a local realization of the response map $p$, if we restrict attention to inputs preceded by some input $u$. The latter $u$ can be chosen to be defined on an arbitrarily small time interval. Note that as $T_u$ is getting smaller, the responses of $p_u$ will get closer to those of $p$. Hence, $p_u$ can be seen as an approximation of $p$ and thus $\Sigma$ can be viewed as an approximation of a Nash realization of $p$. Theorem 5.10 thus indicates that the necessary condition of finite transcendence degree of the observation algebra is in fact very close to being a sufficient one as well. This gives hope that the existence results can be made more sharp and practically more useful.

Note that polynomial systems as defined in [4], see the definition below, are special cases of Nash systems. Hence, sufficient conditions derived in [4] for the existence of polynomial realizations yield sufficient conditions for the existence of Nash realizations. In order to present these conditions and their straightforward application to Nash systems, we first recall the notion of a polynomial system.

\textbf{Definition 5.11} ([4]). A polynomial system $\Sigma$ with an input-space $U \subseteq \mathbb{R}^m$ and an output-space $\mathbb{R}'$ is a quadruple $(X, f, h, x_0)$ where

(i) the state-space $X$ is an irreducible variety in $\mathbb{R}^n$,

(ii) the dynamics of the system is given by $\dot{x}(t) = f(x(t), u(t))$ for an input $u \in U_{\text{pc}}$, where $f : X \times U \rightarrow \mathbb{R}^n$ is such that for every input value $\alpha \in U$ the components $f_{\alpha,i} : X \rightarrow \mathbb{R}$, $i = 1, \ldots, n$ of $f(x, \alpha) = (f_{\alpha,1}(x), \ldots, f_{\alpha,n}(x))$ are polynomial functions on $X$ ($f_{\alpha,i}$ is the $i$th coordinate of the vector field $f_{\alpha} : X \ni x \mapsto f(x, \alpha) \in \mathbb{R}^n$),

(iii) the output of the system is specified by the map $h : X \rightarrow \mathbb{R}'$, the components $h_1, \ldots, h_r$ of $h$ are polynomial functions on $X$,

(iv) $x_0 = x(0) \in X$ is the initial state of $\Sigma$.

The results on the existence of polynomial realizations stated in the following theorem are derived in [4, Theorem 3] and in [41, Theorem 5.2].

\textbf{Theorem 5.12.} Let $p : U_{\text{pc}} \rightarrow \mathbb{R}'$ be a response map. If $A_{\text{obs}}(p)$ is a finitely generated $\mathbb{R}$-algebra then $p$ has a polynomial realization, and hence a Nash realization.
Note that if $A_{\text{obs}}(p)$ is a finitely generated algebra, then its transcendence degree is finite.

5.2. Minimality of Nash systems. By Definition 4.5, a minimal Nash realization of a response map is a Nash realization such that the dimension of its state-space is the smallest one within the class of all Nash realizations of the same response map. As a direct consequence of Theorem 5.8 we can state the following sufficient condition for minimality.

**Theorem 5.13.** If the dimension of a Nash realization $\Sigma$ of a response map $p$ equals $\text{trdeg} \ A_{\text{obs}}(p)$, then $\Sigma$ is a minimal Nash realization of $p$.

Theorem 5.13 is not very appealing, because it does not relate minimality to systems-theoretic properties. However, it has the following corollary, which relates minimality to semi-algebraic reachability and semi-algebraic observability. Before stating this corollary, we state the following result, which is interesting on its own right.

**Theorem 5.14.** A Nash realization $\Sigma$ of a response map $p$ is semi-algebraically reachable and semi-algebraically observable if and only if $\dim(\Sigma) = \text{trdeg} \ A_{\text{obs}}(p)$.

The proof of Theorem 5.14 is presented in Section 7. Theorem 5.14 and Theorem 5.13 then yield the following sufficient condition for minimality.

**Theorem 5.15.** If a Nash realization $\Sigma$ of a response map $p$ is semi-algebraically reachable and semi-algebraically observable then $\Sigma$ is minimal. In particular, if $\Sigma$ is semi-algebraically reachable and strongly semi-algebraically observable, then it is minimal.

Note that both Theorem 5.13 and Theorem 5.15 formulate sufficient conditions for minimality of a Nash realization. At this point we do not know if any Nash realization can be converted to a semi-algebraically reachable and observable one. One way to show this is to show that for any response map $p$, which is realizable by a Nash system, there exists a Nash realization $\Sigma$ such that $\dim(\Sigma) = \text{trdeg} \ A_{\text{obs}}(p)$. We do not know if this is true. However, Theorem 5.10 indicates that this is ‘almost’ the case.

It is natural to wonder what the relationship is between minimality of Nash systems and minimality of analytical non-linear systems; it is clear that if a Nash system is minimal as an analytic system, then it is also minimal as a Nash system. The converse does not hold. Indeed, recall from [16] that a minimal non-linear analytic (smooth) system must be accessible and locally weakly observable. But the system presented in Example 4.19 is neither locally weakly observable nor it is accessible (see Example 4.6), and hence it cannot be minimal as a smooth system. However, it is semi-algebraically reachable and semi-algebraically observable and hence it is minimal as a Nash system (it has the smallest possible dimension among all Nash realizations of its input-output maps).

6. Proofs of the results on the existence of a Nash realization. In this section we present the proofs of the results presented in Subsection 5.1. We start with formulating the concepts of dual input-to-state maps and stating their properties. We then use these properties to present the proofs of Theorem 5.5, Theorem 5.8 and Theorem 5.12.

6.1. Dual input-to-state maps. Below we present the concept of dual input-to-state maps. These maps and their properties will be central to the proofs of the theorems on the existence of Nash realizations. For the same concepts in case of polynomial and rational systems see [4, 3, 28].
DEFINITION 6.1. Let $\Sigma = (X, f, h, x_0)$ be a Nash realization of a response map $p : \overline{U}_{pc} \to \mathbb{R}^r$ and let $x_\Sigma(\cdot; x_0, u)$ be the trajectory of $\Sigma$ corresponding to an input $u \in \overline{U}_{pc}$. The map $\tau_\Sigma : \overline{U}_{pc} \to X$ defined as $\tau_\Sigma(u) = x_\Sigma(T_u; x_0, u)$ for all $u \in \overline{U}_{pc}$ is called the input-to-state map. By the dual input-to-state map of $\Sigma$ we mean the map $\tau_\Sigma^* : \mathcal{N}(X) \to \mathcal{A}(\overline{U}_{pc} \to \mathbb{R})$ such that for every Nash function $g \in \mathcal{N}(X)$ it holds that $\tau_\Sigma^*(g) = g \circ \tau_\Sigma$, i.e.

$$\forall u \in \overline{U}_{pc} : \tau_\Sigma^*(g)(u) = g(\tau_\Sigma(u)) = g(x_\Sigma(T_u; x_0, u)).$$

Intuitively, the dual input-to-state map of a Nash system $\Sigma = (X, f, h, x_0)$ maps each Nash function $g$ on $X$ to the response map which is generated by the Nash system $\Sigma_2 = (X, f, g, x_0)$ where $g$ is used instead of $h$ as the output map. The dual input-to-state map plays a role similar to the observability Grammian of linear systems. In particular, it allows us to relate the properties of a response map $p$ to the properties of the ring of Nash functions on the state-space $X$ of a Nash realization $\Sigma = (X, f, h, x_0)$ of $p$.

PROPOSITION 6.2. Let $\Sigma = (X, f, h, x_0)$ be a Nash realization of a response map $p : \overline{U}_{pc} \to \mathbb{R}^r$ and let $\tau_\Sigma^* : \mathcal{N}(X) \to \mathcal{A}(\overline{U}_{pc} \to \mathbb{R})$ be as in Definition 6.1. Then,

(i) $\forall g \in \mathcal{N}(X) \forall \alpha \in U : D_\alpha \tau_\Sigma^*(g) = \tau_\Sigma^*(L_{f,\alpha}g)$, where $L_{f,\alpha}g$ is the Lie-derivative of the map $g$ with respect to the vector field $f_\alpha : X \ni x \mapsto f(x, \alpha)$, and $D_\alpha$ is the derivation defined in Definition 5.1,

(ii) $A_{obs}(p) = \tau_\Sigma^*(A_{obs}(\Sigma)) \subseteq \tau_\Sigma^*(\mathcal{N}(X))$,

(iii) $\forall g \in \mathcal{N}(\mathbb{R}^k) \forall \varphi_1, \ldots, \varphi_k \in \mathcal{N}(X) \forall u \in \overline{U}_{pc} : \tau_\Sigma^*(g(\varphi_1, \ldots, \varphi_k))(u) = g(\tau_\Sigma^*(\varphi_1)(u), \ldots, \tau_\Sigma^*(\varphi_k)(u)).$

The proof of Proposition 6.2 is analogous to the proof of [4, Lemma 1]. However, [4, Lemma 1] was stated only for polynomial systems and thus it is not directly applicable to our case. Hence, for the sake of completeness, we state the proof of Proposition 6.2 below.

Proof. [Proof of Proposition 6.2]

(i) We show that $\forall g \in \mathcal{N}(X) \forall \alpha \in U$ $\forall u \in \overline{U}_{pc} : D_\alpha \tau_\Sigma^*(g)(u) = \tau_\Sigma^*(L_{f,\alpha}g)(u)$. Let $g \in \mathcal{N}(X)$, $\alpha \in U$, $u \in \overline{U}_{pc}$ be arbitrary. Then,

$$D_\alpha \tau_\Sigma^*(g)(u) =$$

$$= \left. \frac{d}{dt} g(x_\Sigma(T_u + t; x_0, (u))(\alpha, t)) \right|_{t=0+} =$$

$$= \sum_{i=1}^n \left. \frac{d}{dt} g(x_\Sigma(T_u + t; x_0, (u))(\alpha, t)) \frac{d}{dt} x_\Sigma, i(T_u + t; x_0, (u))(\alpha, t) \right|_{t=0+} =$$

$$= \sum_{i=1}^n \left. \frac{d}{dt} g(x_\Sigma(T_u; x_0, u)) f_i(x_\Sigma(T_u; x_0, u), \alpha) = (L_{f,\alpha}g)(x_\Sigma(T_u; x_0, u)) =$$

$$= (L_{f,\alpha}g)(\tau_\Sigma(u)) = \tau_\Sigma^*(L_{f,\alpha}g)(u).$$

(ii) From Definition 4.13, $A_{obs}(\Sigma) \subseteq \mathcal{N}(X)$. To prove (ii) it is sufficient to prove that $A_{obs}(p) = \tau_\Sigma^*(A_{obs}(\Sigma))$.

Because $\Sigma = (X, f, h, x_0)$ is a Nash realization of a response map $p : \overline{U}_{pc} \to \mathbb{R}^r$, it holds that $p_i(u) = h_i(\tau_\Sigma(u))$ for all $u \in \overline{U}_{pc}$ and $i = 1, \ldots, r$. Thus,

$$p_i = \tau_\Sigma^*(h_i) \text{ for all } i = 1, \ldots, r. \quad (6.1)$$
Further, by using (6.1) and repeatedly applying (i) we obtain

$$\forall i = 1, \ldots, r \forall k \in \mathbb{N} \forall \alpha_1, \ldots, \alpha_k \in U : D_{\alpha_k} \cdots D_{\alpha_1} p_i = \tau^\Sigma(L_{f_{\alpha_i}} \cdots L_{f_{\alpha_1}} h_i).$$

Since $A_{obs}(p)$ is generated by the elements of the set \{ $p_i, D_{\alpha_i} \cdots D_{\alpha_1} p_i \mid i = 1, \ldots, r; k \in \mathbb{N}; \alpha_1, \ldots, \alpha_k \in U$ \}, and $A_{obs}(\Sigma)$ is generated by the elements of the set \{ $h_i, L_{f_{\alpha_i}} \cdots \cdots L_{f_{\alpha_1}} h_i \mid i = 1, \ldots, r; k \in \mathbb{N}; \alpha_1, \ldots, \alpha_k \in U$ \}, and $\tau^\Sigma$ is a homomorphism, we can conclude that $A_{obs}(p) = \tau^\Sigma(A_{obs}(\Sigma))$.

(iii) This statement follows directly from the definition of $\tau^\Sigma$. \[ \square \]

**Corollary 6.3.** Let $\Sigma = (X, f, h, x_0)$ be a Nash system such that $\overline{U_{pc}} \subseteq U_{pc}(\Sigma)$ for a set $U_{pc}$ of admissible inputs. Let $\tau^\Sigma : \mathcal{N}(X) \to \mathcal{A}(\overline{U_{pc}} \to \mathbb{R})$ be as in Definition 6.1. Then,

$$\forall g \in \mathcal{N}(X) \forall k \in \mathbb{N} \forall \alpha_1, \ldots, \alpha_k \in U : D_{\alpha_k} \cdots D_{\alpha_1} \tau^\Sigma(g) = \tau^\Sigma(L_{f_{\alpha_k}} \cdots L_{f_{\alpha_1}} g).$$

**Corollary 6.4.** Consider the same assumptions as in Proposition 6.2. Then $\text{trdeg} A_{obs}(p) \leq \text{trdeg} \tau^\Sigma(\mathcal{N}(X))$ and $\text{trdeg} A_{obs}(p) \leq \text{trdeg} A_{obs}(\Sigma)$.

### 6.2. Proof of Theorem 5.5.

**Proof.**

$(\Rightarrow)$ Let $\Sigma = (X, f, h, x_0)$ be a Nash realization of a response map $p : \overline{U_{pc}} \to \mathbb{R}^r$ and assume that $X \subseteq \mathbb{R}^n$. Let $\varphi_i \in \mathcal{A}(\overline{U_{pc}} \to \mathbb{R})$, $i = 1, \ldots, n$ be the maps defined as $\varphi_i(u) = x_{\Sigma,i}(T_u; x_0, u)$ for all $u \in \overline{U_{pc}}$, i.e. $\varphi_i(u)$ is the $i$th component of the state of $\Sigma$ at time $T_u$ under the input $u \in \overline{U_{pc}}$. Let us define the set $\mathcal{A} = \{ \varphi_1, \ldots, \varphi_n \}$. Since $(\varphi_1(u), \ldots, \varphi_n(u)) = x_{\Sigma}(T_u; x_0, u)$ for all $u \in \overline{U_{pc}}$, the condition (i) of the theorem is satisfied.

Because $\Sigma$ is a Nash realization of $p$, $h_i \in \mathcal{N}(X)$ and $p_i(u) = h_i(\varphi_1(u), \ldots, \varphi_n(u))$ for $i = 1, \ldots, r$ and for all $u \in \overline{U_{pc}}$. Hence, by Definition 5.4, $p_i \in \mathcal{A}^{\text{Nash}}(X)$ for all $i = 1, \ldots, r$, i.e. (ii) is valid.

Let $\varphi \in \mathcal{A}^{\text{Nash}}(X)$ and $\alpha \in U$ be arbitrary. To prove (iii) we show that $D_\alpha \varphi \in \mathcal{A}^{\text{Nash}}(X)$. From Definition 5.4 and from the definition of the maps $\varphi_i$, $i = 1, \ldots, n$, there exists $q \in \mathcal{N}(X)$ such that $\varphi = q(\varphi_1, \ldots, \varphi_n) = \tau^\Sigma(q)$. Then, Proposition 6.2(i) implies that

$$D_\alpha \varphi = D_\alpha \tau^\Sigma(q) = \tau^\Sigma(L_{f_\alpha} q)(\varphi_1, \ldots, \varphi_n).$$

As $L_{f_\alpha} q \in \mathcal{N}(X)$, from the last equality above it follows that $D_\alpha \varphi \in \mathcal{A}^{\text{Nash}}(X).

$(\Leftarrow)$ Let $\mathcal{A} = \{ \varphi_1, \ldots, \varphi_n \} \subseteq \mathcal{A}(\overline{U_{pc}} \to \mathbb{R})$ and $X \subseteq \mathbb{R}^n$ be as in the theorem. We will define a Nash system $\Sigma$ realizing $p$. From (ii) and (iii) of the theorem it follows that for all $i = 1, \ldots, r$, $j = 1, \ldots, n$, $\alpha \in U$, and $u \in \overline{U_{pc}}$ there exist $h_i, f_{\alpha,j} \in \mathcal{N}(X)$ such that

$$p_i(u) = h_i(\varphi_1(u), \ldots, \varphi_n(u)) \quad \text{and} \quad D_\alpha \varphi_j(u) = f_{\alpha,j}(\varphi_1(u), \ldots, \varphi_n(u)).$$

Consider the Nash system $\Sigma = (X, f, h, x_0)$ such that for all $x \in X$ and for all $\alpha \in U$ it holds that $f(x, \alpha) = (f_{\alpha,1}(x), \ldots, f_{\alpha,n}(x))$, $h(x) = (h_1(x), \ldots, h_r(x))$ and $x_0 = (\varphi_1(e), \ldots, \varphi_n(e))$. Recall that $e$ stands for the empty input. We prove that this system is a Nash realization of $p$. 

Let \( pr_i : X \to \mathbb{R} : (x_1, \ldots, x_n) \mapsto x_i \) for \( i = 1, \ldots, n \) be the restriction of the projection on the \( i \)th coordinate on \( X \). First, notice that if \( pr_i : X \to \mathbb{R} \) is the \( i \)th projection, i.e. \( pr_i(x_1, \ldots, x_n) = x_i, i = 1, \ldots, n \), then

\[
L_{f_i}pr_i = f_{a,i}.
\]

For every input \( u = (a_1, t_1) \cdots (a_k, t_k) \in \widetilde{U}_{pc} \) we define the map \( \psi_u : [0, T_u] \to X \) as \( \psi_u(t) = (\varphi_1, \ldots, \varphi_n)(u_{[0,t]}) \) for \( t \in [0, T_u] \). Then, for \( i = 1, \ldots, k \) and for \( t \in (\sum_{j=0}^{i-1} t_j, \sum_{j=0}^i t_j) \) (note that \( t_0 = 0 \)) it holds that

\[
\dot{\psi}_u(t) = \frac{d}{dt} \psi_u(t) = \frac{d}{dt} \psi_u(t + \tau)|_{\tau=0^+} = \frac{d}{dt} (\varphi_1(u_{[0,t+\tau]}), \ldots, \varphi_n(u_{[0,t+\tau]}))|_{\tau=0^+}
\]

\[
= \frac{d}{dt} (\varphi_1((a_1, t_1) \cdots (a_{i-1}, t_{i-1}))(a_i, t + \tau - \sum_{j=0}^{i-1} t_j), \ldots, \varphi_n((a_1, t_1) \cdots (a_{i-1}, t_{i-1}))(a_i, t + \tau - \sum_{j=0}^{i-1} t_j))|_{\tau=0^+}
\]

\[
= (D_{a_i}\varphi_1(u_{[0,t]}), \ldots, D_{a_i}\varphi_n(u_{[0,t]}))
\]

\[
= ((L_{f_{a_i}}pr_i)\varphi_1(u_{[0,t]}), \ldots, (L_{f_{a_i}}pr_i)\varphi_n(u_{[0,t]}))
\]

\[
= (f_1(\psi_u(t), a_i), \ldots, f_n(\psi_u(t), a_i)) = f(\psi_u(t), a_i) = f(\psi_u(t), u(t))
\]

and

\[
\psi_u(0) = (\varphi_1(e), \ldots, \varphi_n(e)) = x_0.
\]

Therefore, by Definition 3.3, \( \psi_u \) is the state trajectory of \( \Sigma \) corresponding to the input \( u \), i.e. \( \psi_u(t) = x_\Sigma(t; x_0, u) \) for \( t \in [0, T_u] \). Hence, for arbitrary \( u \in \widetilde{U}_{pc} \) and for all \( i = 1, \ldots, r \),

\[
h_i(x_\Sigma(T_u; x_0, u)) = h_i(\psi_u(T_u)) = h_i((\varphi_1, \ldots, \varphi_n)(u)) = h_i(\varphi_1(u), \ldots, \varphi_n(u)) = p_i(u).
\]

Thus, \( \Sigma \) is indeed a Nash realization of \( p \). \( \square \)

### 6.3. Proofs of Theorems 5.8 and 5.10.

**Proof.** Let \( \Sigma = (X, f, h, x_0) \) be a Nash realization of a response map \( p : \widetilde{U}_{pc} \to \mathbb{R}^r \). By Proposition 2.8, \( N(X) \) is algebraic over \( \mathbb{R}[X] \) and \( \text{trdeg } N(X) = \text{trdeg } \mathbb{R}[X] = \dim(X) \). By applying Corollary 6.4, we derive that \( \text{trdeg } A_{obs}(p) \leq \text{trdeg } \tau(X) \leq \text{trdeg } N(X) = \dim(X) \). \( \square \)

**Proof.** [Proof of Theorem 5.10] Assume that \( \varphi_1, \varphi_2, \ldots, \varphi_d \in A_{obs}(p) \) is a transcendence basis of \( A_{obs}(p) \). Then there exist polynomials \( Q_{\alpha,i}(Y, X_1, \ldots, X_d), i = 1, 2, \ldots, d, \alpha \in U \) and \( P_j(Y, X_1, \ldots, X_d), j = 1, 2, \ldots, r \) in \( d + 1 \) variables such that

\[
Q_{\alpha,i}(D_\alpha \varphi_1, \varphi_2, \ldots, \varphi_d) = 0 \text{ for all } \alpha \in U, i = 1, 2, \ldots, d
\]

\[
P_j(p_j, \varphi_1, \varphi_2, \ldots, \varphi_d) = 0 \text{ for all } j = 1, 2, \ldots, r
\]

(6.2)

By choosing \( Q_{\alpha,i} \) and \( P_j \) such that they are of minimal degree with respect to \( Y \), we can assume that \( \frac{1}{\tau} Q_{\alpha,i}(D_\alpha \varphi_1, \varphi_2, \ldots, \varphi_d) \neq 0 \) and \( \frac{1}{\tau} P_j(p_j, \varphi_1, \varphi_2, \ldots, \varphi_d) \neq 0 \)
viewed as elements of \( \mathcal{A} \) \((\mathcal{U}_{pc} \to \mathbb{R}) \). Let \( R \) be the product of \( \frac{\partial}{\partial Y} Q_{\alpha,i}(D_\alpha \varphi_i, \varphi_1, \varphi_2, \ldots, \varphi_d)'s \) and \( \frac{\partial}{\partial Y} P_j(p_j, \varphi_1, \varphi_2, \ldots, \varphi_d)'s \), i.e.

\[
R = (\prod_{i \in U} \prod_{j=1}^d \frac{d}{dY} Q_{\alpha,i}(D_\alpha \varphi_i, \varphi_1, \varphi_2, \ldots, \varphi_d))(\prod_{j=1}^d \frac{d}{dY} P_j(p_j, \varphi_1, \varphi_2, \ldots, \varphi_d)).
\]

Clearly, \( R \) is a response map and as \( \mathcal{A} \) \((\mathcal{U}_{pc} \to \mathbb{R}) \) is an integral domain, \( R \neq 0 \), since none of the factors of \( R \) is zero. Hence, there exists \( v \in \mathcal{U}_{pc} \) such that \( R(v) \neq 0 \). Assume that \( v = (\beta_1, s_1) \cdots (\beta_t, s_t), \beta_1, \ldots, \beta_t \in U, \ s_1, \ldots, s_t \in [0, +\infty) \). Then, from the properties of admissible inputs it follows that there exists a connected open subset \( V \) of \([0, +\infty)^t \) such that \((0, \ldots, 0), (s_1, \ldots, s_t) \in V \) and for any \((\tau_1, \ldots, \tau_t) \in V, \ (\beta_1, \tau_1) \cdots (\beta_t, \tau_t) \in \mathcal{U}_{pc} \) and the map

\[
g : V \ni (\tau_1, \ldots, \tau_t) \mapsto R((\beta_1, \tau_1) \cdots (\beta_t, \tau_t))
\]

is analytic. Since \( g(s_1, \ldots, s_t) \neq 0 \), it then follows the set \( S \) of those elements \( \tau \in V \) such that \( g(\tau) = 0 \) is closed and its interior is empty. Hence, for any \((\tau_1, \ldots, \tau_t) \in V \setminus S, u = (\beta_1, \tau_1) \cdots (\beta_t, \tau_t), R(u) \neq 0 \) and thus \( \frac{\partial}{\partial Y} Q_{\alpha,i}(D_\alpha \varphi_i, \varphi_1, \varphi_2, \ldots, \varphi_d(u)) \neq 0 \) and \( \frac{\partial}{\partial Y} P_j(p_j(u), \varphi_1, \varphi_2, \ldots, \varphi_d(u)) \neq 0 \). Note that since \( 0 \in V \) and \( S \) has empty interior, for any \( \epsilon > 0 \), there exists a choice \((\tau_1, \ldots, \tau_k)\), such that \( T_u = \sum_{i=1}^k \tau_i < \epsilon \).

We apply now the implicit function theorem \([8, Corollary 2.9.8]\) to \( Q_{\alpha,i} \) and \( P_j \) at points \((y_{0,i}^0, x^0) = (D_\alpha \varphi_i(u), \varphi_1(u), \varphi_2(u), \ldots, \varphi_d(u)) \) and \((y_j^0, x^0) = (p_j(u), \varphi_1(u), \varphi_2(u), \ldots, \varphi_d(u)) \), where \( x^0 = (\varphi_1(u), \ldots, \varphi_d(u)) \). It then follows that there exist a semi-algebraic open subset \( X \subseteq \mathbb{R}^d \) and Nash maps \( f_{\alpha,i}, h_j \in \mathcal{N}(X), i = 1, \ldots, d, \) and semi-algebraic open subsets \( V_{\alpha,i}, V_j \) of \( U = \mathbb{R}^r \) such that

1. \( x^0 \in X, y_{0,i}^0 \in V_{\alpha,i}, y_j^0 \in V_j, \alpha \in U, i = 1, \ldots, d, j = 1, \ldots, r, \) and
2. for every \((y_{0,i}, x) \in V_{\alpha,i} \times X, (y_j, x) \in V_j \times X, P_j(y_j, x) = 0 \iff y_j = h_j(x) \) and \( Q_{\alpha,i}(y_{\alpha,i}, x) = 0 \iff y_{\alpha,i} = f_{\alpha,i}(x), \alpha \in U, i = 1, \ldots, d, j = 1, \ldots, r. \)

We will show that \( \Sigma = (X, f, h, x^0) \), where \( f(x, \alpha) = (f_{\alpha,1}(x), \ldots, f_{\alpha,d}(x))^T \) and \( h(x) = (h_1(x), \ldots, h_r(x)) \), is a local realization of \( p_u \).

To this end, take an arbitrary sequence of inputs \( a_1, \ldots, a_k \in U, k \geq 0 \). For any \( \ell = (t_1, \ldots, t_k) \in [0, +\infty)^k \), denote by \( v(\ell) \) the input \((a_1, t_1) \cdots (a_k, t_k)\). From the properties of admissible inputs it follows that there exists an open subset \( W \subseteq [0, +\infty)^k, 0 \in W \) such that for all \( \ell \in W, uv(\ell) \in \mathcal{U}_{pc} \). Define \( G : W \ni \ell \mapsto (\varphi_1(uv(\ell), \ldots, \varphi_d(uv(\ell))) \) and \( G_{\alpha,i} : W \ni \ell \mapsto D_\alpha \varphi_i(uv(\ell)) \) and \( H_j : W \ni \ell \mapsto p_j(uv(\ell)) \). Notice \( G(0) = x^0, G_{\alpha,0}(0) = y_{0,i}^0 \) and \( H_j(0) = y_j^0, i = 1, \ldots, d, j = 1, \ldots, r, \alpha \in U \). Hence, there exists a subset \( W' \) of \( W \) of the form \( W' = [0, r]^k \) for some \( r > 0 \), such that for all \( \ell \in W' \), \( G(\ell) \subseteq X, G_{\alpha,i}(\ell) \in V_{\alpha,i}, Q_{\alpha,i}(G_{\alpha,i}(\ell), G(\ell)) = 0 \) and \( H_j(\ell) \in V_j, P_j(H_j(\ell), G(\ell)) = 0, i = 1, \ldots, d, j = 1, \ldots, r, \alpha \in U \). Hence, \( H_j(\ell) = h_j(G(\ell)) \) and \( G_{\alpha,i}(\ell) = f_{\alpha,i}(G(\ell)) \) for all \( \ell \in W' \) and \( i = 1, \ldots, d, j = 1, \ldots, r, \alpha \in U \).

For every \( \ell = (t_1, \ldots, t_k) \in W', \) define \( x_\Sigma(\ell; x^0, v(\ell)) : [0, \sum_{i=1}^k t_i] \to X \) as follows:

\[
x_\Sigma(\ell; x^0, v(\ell)) = G((t_1, \ldots, t_i, t - \sum_{j=1}^{i-1} t_j, 0, \ldots, 0)) \text{ if } t \in (\sum_{j=1}^i t_j, \sum_{j=1}^{i+1} t_j) \text{ for some } i = 0, \ldots, k-1.
\]

Note that if \((t_1, \ldots, t_k) \in W' \), then \((t_1, \ldots, t_i, s, 0, \ldots, 0) \in W' \) for \( s < t_{i+1}, i = 0, \ldots, k-1 \). Notice that the \( l \)th entry of the vector \( \frac{\partial}{\partial x} x_\Sigma(\ell; x^0, v(\ell)) \) equals \( G_{\alpha,i}(t_1, \ldots, t_i, t - \sum_{j=1}^{i+1} t_j) \) if \( t \in (\sum_{j=1}^i t_j, \sum_{j=1}^{i+1} t_j] \). Hence, \( x(t) = x_\Sigma(t; x^0, v(\ell)) \) satisfies the differential equation \( \dot{x}(t) = f(x(t), v(\ell)(t)) \) with initial state \( x(0) = x^0 \). It follows from the definitions above that for any \( \ell \in W' \),

\[
p(uv(\ell)) = h(x_\Sigma(T_{v(\ell)}; x, v(\ell))).
\]
Since the choice of $\alpha_1, \ldots, \alpha_k$ was arbitrary, it follows that $\Sigma$ is indeed a local realization of $p_u$. \qed

7. Proof of the results on minimality of Nash realizations. In this section we present the proof of the sufficient condition for the minimality presented in Subsection 5.2. To this end, we need the following technical results characterizing semi-algebraic reachability.

**Proposition 7.1.** Let $\Sigma = (X, f, h, x_0)$ be a Nash realization of a response map $p : \widetilde{U}_{pc} \to \mathbb{R}^r$. Consider the following statements.

(i) $\Sigma$ is semi-algebraically reachable,
(ii) the dual input-to-state map $\tau^*_\Sigma : N(X) \to A(\widetilde{U}_{pc} \to \mathbb{R})$ is injective,
(iii) the ideal $\text{Ker} \, \tau^*_\Sigma$ is the zero ideal in $N(X)$.
(iv) $\text{trdeg} \, A_{\text{obs}}(p) = \text{trdeg} \, A_{\text{obs}}(\Sigma)$.

The statements (i), (ii) and (iii) are equivalent. The statement (ii) implies (iv). If $\Sigma$ is semi-algebraically observable, then (iv) and (ii) are equivalent.

**Proof.** The equivalence (ii) $\iff$ (iii) and the implication (ii) $\implies$ (iv) are obvious. If $\Sigma$ is semi-algebraically observable, then $\text{trdeg} \, N(X) = \text{trdeg} \, A_{\text{obs}}(\Sigma)$. Hence, if $\text{trdeg} \, A_{\text{obs}}(\Sigma) = \text{trdeg} \, A_{\text{obs}}(p)$, then $\text{trdeg} \, N(X) = \text{trdeg} \, \tau^*_\Sigma(N(X)) < +\infty$. From [42, Chapter II, Theorem 28,29] it follows that the latter is possible only if $\tau^*_\Sigma$ is injective.

It is left to prove (i) $\iff$ (iii).

By Definition 4.7 and 4.8, $\Sigma$ is semi-algebraically reachable if and only if

$$\forall g \in N(X) : \left( \forall u \in \widetilde{U}_{pc} : g(x_{\Sigma}(T_u; x_0, u)) = 0 \right) \Rightarrow g = 0.$$ 

Using the notation of $\tau^*_\Sigma$, see Definition 6.1, one can reformulate this characterization as $\forall g \in N(X) : \tau^*_\Sigma(g) = 0 \Rightarrow g = 0$. Thus, $\Sigma$ is semi-algebraically reachable if and only if $\text{Ker} \, \tau^*_\Sigma = (0) \subseteq N(X)$.

**Proof.** [Proof of Theorem 5.14] Let $\Sigma = (X, f, h, x_0)$ be a semi-algebraically reachable and semi-algebraically observable Nash realization of a response map $p$. By Proposition 2.8, $\dim(\Sigma) = \dim(X) = \text{trdeg} \, N(X)$. Then, because $\Sigma$ is semi-algebraically observable, see Definition 4.14,

$$\dim(\Sigma) = \text{trdeg} \, A_{\text{obs}}(\Sigma). \quad (7.1)$$

Since $\Sigma$ is semi-algebraically reachable, it follows by Proposition 7.1 that

$$\text{trdeg} \, A_{\text{obs}}(\Sigma) = \text{trdeg} \, A_{\text{obs}}(p). \quad (7.2)$$

Finally, by (7.1) and (7.2), $\dim(\Sigma) = \text{trdeg} \, A_{\text{obs}}(p)$.

Conversely, assume that $\dim(\Sigma) = \text{trdeg} \, A_{\text{obs}}(p)$. If $\Sigma$ is not semi-algebraically observable, then $\text{trdeg} \, A_{\text{obs}}(\Sigma) < \dim(\Sigma)$. Since $\tau^*_\Sigma(A_{\text{obs}}(\Sigma)) = A_{\text{obs}}(p)$, it follows that $\text{trdeg} \, A_{\text{obs}}(p) \leq \text{trdeg} \, A_{\text{obs}}(\Sigma) < \dim(\Sigma)$, which is a contradiction. Hence, $\Sigma$ must be semi-algebraically observable. From Proposition 7.1 and $\text{trdeg} \, A_{\text{obs}}(p) = \dim(\Sigma) = \text{trdeg} \, A_{\text{obs}}(\Sigma)$ it then also follows that $\Sigma$ is semi-algebraically reachable. \qed

8. Conclusions. In this paper we have introduced the class of Nash systems and we have formulated the realization problem for it. Our motivation for studying Nash systems is their relevance both for theory and applications. The presented results on realization theory of Nash systems are not yet complete.

Further research aims at extending these results to necessary and sufficient conditions for the existence of minimal Nash realizations, characterizing the relations.
between Nash realizations of the same response maps, and obtaining a realization algorithm. In addition, we would like to investigate the applications of the results of the paper to system identification, model reduction, observer and control design of Nash systems. Another possible research direction is to explore the extension of our results to systems which are defined on Nash manifolds with boundaries.

Acknowledgment The authors thank the anonymous referees for their numerous useful remarks and suggestions.

REFERENCES

Appendix A. Proofs of the technical results.

Proof. [Proof of Proposition 2.8] Let \( g(x) = q(f_1(x), \ldots, f_k(x)) \), \( x \in X \) be as in Proposition 2.8. We prove that \( g \) is algebraic over \( \mathbb{R}[f_1, \ldots, f_k] \).

By Lemma 2.1, there exist semi-algebraic subsets \( S_1, \ldots, S_d \) of \( S \) and polynomials \( g_i \in \mathbb{R}[X_1, \ldots, X_k, Y] \), \( i = 1, \ldots, d \) such that \( S = \bigcup_{i=1}^d S_i, S_i \cap S_j = \emptyset \) for all \( i \neq j \) and such that for all \( s \in S_i \), \( g_i(s, Y) \neq 0 \) and \( g_i(s, q(s)) = 0 \).

For every \( x \in X \) there exists a unique index \( i(x) \in \{1, \ldots, d\} \) such that \( \xi(x) = (f_1(x), \ldots, f_k(x)) \in S_{i(x)} \). Indeed, \( \xi(x) \in S \), and hence \( \xi(x) \) belongs to one of the disjoint sets \( S_{i(x)} \). Then \( g_i(\xi(x), Y) \neq 0 \) and \( g_i(\xi(x), q(\xi(x))) = g_i(\xi(x), q(x)) = 0 \). Consider the subset \( I = \{i(x)|x \in X\} \subseteq \{1, \ldots, d\} \). Assume that \( I = \{i_1, \ldots, i_j\} \).

Consider the polynomial \( P = g_{i_1} \cdots g_{i_j} \in \mathbb{R}[X_1, \ldots, X_k, Y] \). Then \( P(\xi(x), q(x)) = 0 \) for all \( x \in X \). The proof is complete once we show \( P(\xi(x), Y) \neq 0 \) for some \( x \in X \).

The fact that there exists \( x \in X \) such that \( P(\xi(x), Y) \neq 0 \) is equivalent to the requirement that the polynomial \( P(f_1, \ldots, f_k, Y) \in \mathcal{N}(X)[Y] \) with coefficients in \( \mathcal{N}(X) \) is not identically zero. Since \( \mathcal{N}(X) \) is an integral domain, it holds that the algebra \( \mathcal{N}(X)[Y] \) is also an integral domain, see [42, Vol.1, Ch.1.16]. Hence, if \( P(f_1, \ldots, f_k, Y) = \Pi_{i=1}^n g_i(f_1, \ldots, f_k, Y) \), where \( g_i(f_1, \ldots, f_k, Y) \in \mathcal{N}(X)[Y] \), equals zero as an element of \( \mathcal{N}(X)[Y] \) then there exists at least one \( i \in I \) such that \( g_i(f_1, \ldots, f_k, Y) = 0 \). The latter means that \( g_i(f_1(x), \ldots, f_k(x), Y) = g_i(\xi(x), Y) = 0 \) for all \( x \in X \). But there exists at least one \( \hat{x} \) such that \( i = i(\hat{x}) \), i.e. such that
ξ(\dot{x}) \in S_i \text{ and } g_i(ξ(\dot{x}),Y) \neq 0 \text{ which is a contradiction. Therefore, we get that } P(f_1, \ldots, f_n,Y) \neq 0.

To prove that the ring \mathcal{N}(X) is algebraic over the polynomial ring \mathbb{R}[X] we consider arbitrary \ f \in \mathcal{N}(X) \text{ and repeat the proof above with }
\begin{itemize}
  \item \ f_i = pr_i, i = 1, \ldots, n, \text{ where } pr_i : X \rightarrow \mathbb{R} : x \mapsto x_i \text{ is the restriction of the projection map on the } i\text{th coordinate to } X,
  \item S = X,
  \item g = g = f.
\end{itemize}
\end{proof}

Next, we present the proof of Proposition 4.18. To this end, we need the following auxiliary result.

**Proposition A.1.** Let Σ = (X,f,h,x_0) be a Nash system. Then A_{obs}^{Nash}(Σ) is algebraic over A_{obs}(Σ) and consequently trdeg A_{obs}^{Nash}(Σ) = trdeg A_{obs}(Σ).

**Proof.** Consider an arbitrary \ g \in A_{obs}^{Nash}(Σ). According to Definition 4.16 there exist \ k \in \mathbb{N}, \varphi_1, \ldots, \varphi_k \in A_{obs}(Σ), \text{ and } q \in \mathcal{N}(S) \text{ for some Nash manifold } S \subseteq \mathbb{R}^k, \text{ such that }
\begin{equation}
g(x) = q(\varphi_1(x), \ldots, \varphi_k(x)) \text{ for all } x \in X.
\end{equation}

From Proposition 2.8 it follows that \ g \ is algebraic over \ \mathbb{R}[\varphi_1, \ldots, \varphi_k] \subseteq A_{obs}(Σ). Because \ g \in A_{obs}^{Nash}(Σ) \text{ was arbitrary, } A_{obs}^{Nash}(Σ) \text{ is algebraic over } A_{obs}(Σ). This implies that trdeg A_{obs}^{Nash}(Σ) = trdeg A_{obs}(Σ).

**Proof.** [Proof of Proposition 4.18] As Σ is strongly semi-algebraically observable, we derive from Definition 4.17 that \mathcal{N}(X) = A_{obs}^{Nash}(Σ). Consequently, trdeg \mathcal{N}(X) = trdeg A_{obs}^{Nash}(Σ). Then, from Proposition A.1, trdeg \mathcal{N}(X) = trdeg A_{obs}(Σ) which proves that Σ is semi-algebraically observable.

**Proof.** [Proof of Proposition 4.20] Let dim(X) = d and assume that \ X \subseteq \mathbb{R}^n. Let \varphi_1, \ldots, \varphi_d \text{ be a transcendence basis of } A_{obs}(Σ). Let x_1, \ldots, x_n \text{ be the coordinate functions of } R[X], \text{ i.e. } \pi_i(x) \text{ yields the } i\text{th coordinate of a point } x \in \mathbb{R}^n, \text{ then } x_i \text{ is the restriction of } \pi_i \text{ to } X. \text{ It is clear that } \pi_1, \ldots, \pi_n \in \mathcal{N}(X). \text{ Since } A_{obs}(Σ) \subseteq \mathcal{N}(X) \text{ and trdeg } A_{obs}(Σ) = trdeg \mathcal{N}(X), \text{ it then follows that } \pi_1, \ldots, \pi_n \text{ are algebraic over } A_{obs}(Σ), \text{ i.e. there exist non-zero polynomials } Q_1, \ldots, Q_n \in \mathbb{R}[T_1, T_2, \ldots, T_d] \text{ such that } Q_i(\pi_1, \varphi_1, \ldots, \varphi_d) = 0 \text{ for all } x \in X, \text{ i.e. } x = (x_1, \ldots, x_n), \text{ where } Q_i(\pi_1, \varphi_1, \ldots, \varphi_d) \text{ is viewed as a map } X \ni x \mapsto Q_i(\pi_1(x), \varphi_1(x), \ldots, \varphi_d(x)). \text{ The latter map clearly belongs to } \mathcal{N}(X). \text{ By choosing } Q_1, \ldots, Q_n \text{ to be of minimal degree with respect to the indeterminant } T, \text{ we can assume that } \frac{\partial}{\partial \pi_i} Q_i(\pi_1, \varphi_1, \ldots, \varphi_d) \neq 0, \text{ i.e. } 1, \ldots, n. \text{ Consider the product } R = \frac{\partial}{\partial \pi_i} Q_i(\pi_1, \varphi_1, \ldots, \varphi_d) \ldots \frac{\partial}{\partial \pi_i} Q_n(\pi_1, \varphi_1, \ldots, \varphi_d) \text{ and notice that it can be viewed as an element of } \mathcal{N}(X). \text{ Since } \mathcal{N}(X) \text{ is integral domain, it then follows that } R \neq 0 \text{ and hence there exists } x \in X \text{ such that } R(x) \neq 0. \text{ In other words, there exists } x \in X \text{ such that } Q_i(\pi_1(x), \varphi_1(x), \ldots, \varphi_d(x)) = 0 \text{ and } \frac{\partial}{\partial \pi_i} Q_i(\pi_1(x), \varphi_1(x), \ldots, \varphi_d(x)) \neq 0 \text{ for all } i = 1, \ldots, n. \text{ Define } G : \mathbb{R}^{n+d} \rightarrow \mathbb{R}^n \text{ by } G(y,z) = (G_1(y,z), \ldots, G_n(y,z)), \text{ where } G_i(y_1, \ldots, y_n, z) = Q_i(y_i, z), \text{ i.e. } 1, \ldots, n \text{ and let } y_0 = (\pi_1(x), \ldots, \pi_n(x)) = x \text{ and } z_0 = (\varphi_1(x), \ldots, \varphi_d(x)). \text{ It then follows that the matrix } \frac{\partial}{\partial z_j} G(y_0, z_0) \text{ is invertible. Hence, from the implicit function theorem } [8, \text{ Corollary 2.9.8}] \text{ it follows that there exist open semi-algebraic subsets } W_1, W_2 \text{ of } \mathbb{R}^n \text{ and } \mathbb{R}^d \text{ respectively and a Nash map } \psi : W_2 \rightarrow W_1 \text{ such that } z_0 \in W_2, y_0 \in W_1 \text{ and for any } (y,z) \in W_1 \times W_2, G(y,z) = 0 \iff y = \psi(z). \text{ Define } V = \{ s \in X \mid s \in W_1 \text{ and } (\varphi_1(s), \ldots, \varphi_d(s)) \in W_2 \}. \text{ Then } V \text{ is an open semi-algebraic subset of } X \text{ and } x \in V. \text{ Moreover, for any } s \in V, G(s, \varphi_1(s), \ldots, \varphi_d(s)) = 0 \text{ and hence } s = (\varphi_1(s), \ldots, \varphi_d(s)) \text{ for all } s \in V.
If $V$ is not semi-algebraically connected, then we replace $V$ by the biggest semi-
algebraically connected component of $V$ which contains $x$.

Notice that since $V$ is a semi-algebraic and open subset of $X$, $V$ is also a Nash
submanifold of dimension $d$. If $g \in N(V)$, then $g(s) = g(\psi(\varphi_1(s), \ldots, \varphi_d(s)))$ for
all $s \in V$. Take then $S = W_2 \cap \psi^{-1}(V)$; since $S$ is open subset of $\mathbb{R}^d$, it is a $d$-
dimensional Nash submanifold. It then follows that $g = g \circ \psi \in N(S)$ and $g(s) =
g(\varphi_1(s), \ldots, \varphi_d(s))$, for all $s \in V$. $\square$

Proof. [Proof of Proposition 4.21] Let $\Sigma = (X, f, h, x_0)$ be a strongly semi-
algernably observable Nash system. Because Nash functions on $X$ distinguish
the points of $X$, it holds that all states of $\Sigma$ are distinguishable by the elements
of $A_{\text{obs}}^{\text{Nash}}(\Sigma) = \mathcal{N}(X)$. That is,

$$\forall x_1, x_2 \in X, x_1 \neq x_2 \exists g \in A_{\text{obs}}^{\text{Nash}}(\Sigma) : g(x_1) \neq g(x_2).$$

We prove that the same statement holds even if we can choose $g$ only from $A_{\text{obs}}(\Sigma)$.

Let us assume that for some $x_1 \neq x_2 \in X$ there exists only $g \in A_{\text{obs}}^{\text{Nash}}(\Sigma) \setminus A_{\text{obs}}(\Sigma)$
which distinguishes $x_1$ and $x_2$. Let $g$ be such a map, i.e. $g \in \mathcal{N}(X)$ and $g(x_1) \neq g(x_2)$.

Then, by Definition 4.16, there exist $k \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_k \in A_{\text{obs}}(\Sigma)$, and $q \in \mathcal{N}(S)$
for some Nash manifold $S \subseteq \mathbb{R}^k$ such that

$$g(x_1) = q(\varphi_1(x_1), \ldots, \varphi_k(x_1)) \neq q(\varphi_1(x_2), \ldots, \varphi_k(x_2)) = g(x_2).$$

If $\varphi_i(x_1) = \varphi_i(x_2)$ for all $i = 1, \ldots, k$ then it would imply that $q(\varphi_1(x_1), \ldots, \varphi_k(x_1)) =
q(\varphi_1(x_2), \ldots, \varphi_k(x_2))$. Therefore, since $g(x_1) \neq g(x_2)$, it follows that there is at least
one $i \in \{1, \ldots, k\}$ such that $\varphi_i(x_1) \neq \varphi_i(x_2)$. Thus, $\varphi_i \in A_{\text{obs}}(\Sigma)$ distinguishes $x_1$
and $x_2$ which is a contradiction. $\square$

Proof. [Proof of Corollary 4.22] Consider arbitrary $x_1 \neq x_2 \in X$. We denote by $\tau_{\Sigma, x_1}$ and by $\tau_{\Sigma, x_2}$
the dual input-to-state maps corresponding to the Nash system
$\Sigma = (X, f, h)$ with the initial state $x_1$ and $x_2$, respectively. Denote by $\tilde{U}_{\text{pc}}(x_1, x_2)$ the
set of all piecewise-constant inputs $u$ such that both $x_{\Sigma}(\cdot; x_1, u)$ and $x_{\Sigma}(\cdot; x_2, u)$
are defined. It is easy to see that $\tilde{U}_{\text{pc}}(x_1, x_2)$ is a set of admissible inputs. Let us assume by
contradiction that $h(x_{\Sigma}(T_u; x_1, u)) = h(x_{\Sigma}(T_u; x_2, u))$ for all $u \in \tilde{U}_{\text{pc}}(x_1, x_2)$. Then,

$$D_{\alpha_1} \cdots D_{\alpha_r} \tau_{\Sigma, x_1}^* (h_i) = D_{\alpha_1} \cdots D_{\alpha_r} \tau_{\Sigma, x_2}^* (h_i)$$

(A.1)

for $i = 1, \ldots, r$ and for all $k \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_k \in U$. Because $\tilde{U}_{\text{pc}}(x_1, x_2)$ is a set of
admissible inputs and because all components $h_i$, $i = 1, \ldots, r$ of $h$ are Nash functions
on $X$, we derive by Corollary 6.3 that

$$D_{\alpha_1} \cdots D_{\alpha_j} \tau_{\Sigma, x_j}^* (h_i) = \tau_{\Sigma, x_j}^* (L_{f_{\alpha_1}} \cdots L_{f_{\alpha_j}} h_i)$$

(A.2)

for $j = 1, 2$ and for all $i = 1, \ldots, r$, $k \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_k \in U$. From (A.1) and (A.2),

$$\tau_{\Sigma, x_1}^* (L_{f_{\alpha_1}} \cdots L_{f_{\alpha_j}} h_i) = \tau_{\Sigma, x_2}^* (L_{f_{\alpha_1}} \cdots L_{f_{\alpha_j}} h_i)$$

for $i = 1, \ldots, r$, $k \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_k \in U$. Thus, $L_{f_{\alpha_1}} \cdots L_{f_{\alpha_j}} (h_i) =
L_{f_{\alpha_1}} \cdots L_{f_{\alpha_j}} (h_i) (x_{\Sigma}(T_u; x_1, u)) = L_{f_{\alpha_1}} \cdots L_{f_{\alpha_j}} (h_i) (x_{\Sigma}(x_1, u))$

for all $u \in \tilde{U}_{\text{pc}}(x_1, x_2)$ and hence also for the empty input $e \in \tilde{U}_{\text{pe}}(x_1, x_2)$. Then,

$$L_{f_{\alpha_1}} \cdots L_{f_{\alpha_j}} (h_i)(x_1) = L_{f_{\alpha_1}} \cdots L_{f_{\alpha_j}} (h_i)(x_2)$$

(A.3)

for all $i = 1, \ldots, r$, $k \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_k \in U$. Because $\mathbb{R}[\{L_{f_{\alpha_1}} \cdots L_{f_{\alpha_j}} h_i \mid i =
1, \ldots, r, k \in \mathbb{N}, \alpha_1, \ldots, \alpha_k \in U\}] = A_{\text{obs}}(\Sigma)$, (A.3) implies that $\forall g \in A_{\text{obs}}(\Sigma) : g(x_1) =
g(x_2)$ which contradicts the fact that $\Sigma$ is strongly semi-algebraically observable.
Therefore, there exists \( u \in U_{pc}(x_1, x_2) \) such that \( h(x_\Sigma(T_u; x_1, u)) \neq h(x_\Sigma(T_u; x_2, u)) \).

\[ \Box \]

**Proof.** [Proposition 4.23] If \( \Sigma \) is semi-algebraically observable, then by Proposition 4.20 the restriction \( \Sigma_V \) of \( \Sigma \) to some semi-algebraic subset \( V \) of \( X \) is strongly semi-algebraically observable. From Corollary 4.22 it follows that \( \Sigma_V \) is observable, in particular, it is weakly observable. Then from [7, Theorem 2.1] it follows that \( \Sigma_V \), being an analytic system, is also locally weakly observable. Notice that the observability co-distribution of \( \Sigma_V \) is just the restriction of \( \Delta \) to \( V \). Hence, according to [15, Theorem 3.11], there exists an open subset \( W \) of \( V \) such that for all \( x \in W : \dim(\Delta(x)) = \dim(V) = \dim(X). \)

If \( \dim(\Delta(x)) = \dim(X) \) for some \( x \in X \), then there exists an open semi-algebraic subset \( V \) of \( X \), such that \( x \in V \) and \( \dim(\Delta(z)) = \dim(X) \) for all \( z \in V \). Hence, there exist \( \varphi_1, \ldots, \varphi_d \in A_{obs}(\Sigma) \), such that for a \( d\varphi_1(z), \ldots, d\varphi_d(z) \) are linearly independent for all \( z \in V \). Consider the map \( g : V \ni z \mapsto (\varphi_1(z), \ldots, \varphi_d(z)) \). By the inverse function theorem, there exists an open semi-algebraic sets \( U_1 \subseteq V \) and \( U_2 \subseteq \mathbb{R}^d \) such that \( g : U_1 \to U_2 \) is a Nash diffeomorphism. Assume now that \( \varphi_1, \ldots, \varphi_d \) are not algebraically independent. Then there exists a non-zero polynomial \( Q \) such that \( Q(g(x)) = 0 \) for all \( x \in U_1 \). But it then follows that \( Q(g(g^{-1}(y))) = Q(y) = 0 \) for all \( y \in U_2 \) and hence \( Q \) is zero on \( U_2 \subseteq \mathbb{R}^d \). Hence, \( Q \) must be a zero polynomial, which is a contradiction. But then it follows that \( \varphi_1, \ldots, \varphi_d \) are algebraically independent and hence \( \text{trdeg } A_{obs}(\Sigma) = \dim(X) = \text{trdeg } \mathcal{N}(X) \), i.e. \( \Sigma \) is semi-algebraically observable.

\[ \Box \]