Infinite horizon control and minimax observer design for linear DAEs

Sergiy Zhuk† and Mihaly Petreczky‡
† IBM Research, Dublin, Ireland, sergiy.zhuk@ie.ibm.com,
‡ Ecole des Mines de Douai, Douai, France, mihaly.petreczky@mines-douai.fr

Abstract—In this paper we construct an infinite horizon minimax state observer for a linear stationary differential-algebraic equation (DAE) with uncertain but bounded input and noisy output. We do not assume regularity or existence of a (unique) solution for any initial state of the DAE. Our approach is based on a generalization of Kalman’s duality principle. In addition, we obtain a solution of infinite-horizon linear quadratic optimal control problem for DAE.

I. INTRODUCTION

Consider a linear Differential-Algebraic Equation (DAE) with state $x$, output $y$ and noises $f$ and $\eta$:

$$
\frac{d(Fx)}{dt} = Ax(t) + f(t), \quad Fx(t_0) = x_0,
$$

$$
y(t) = Hx(t) + \eta(t)
$$

where $F, A \in \mathbb{R}^{n \times n}$, $H \in \mathbb{R}^{p \times n}$. We do not restrict DAE’s coefficients, in particular, we do not require that it has a solution for any initial condition $x_0$ or that this solution is unique. The only assumption we impose is that $x_0, f$ and $\eta$ are uncertain but bounded and belong to an ellipsoid in $L^2$. We will consider only solutions which are locally integrable functions. We would like to estimate a state component $\ell^T Fx(t), \ell \in \mathbb{R}^n$ of the DAE based on the output $y$. The desired observer should be linear in $y$, i.e. we are looking for maps $U(t, \cdot) \in L^2$ such that the estimate of $\ell^T Fx(t)$ at time $t$ is of the form $\int_0^t U(t, s)y(s)ds$. The goal of the paper is to find an observer $U$ such that:

1) The worst-case asymptotic observation error $\limsup_{t \to \infty} \sup_{f,\eta}(\ell^T Fx(t) - \int_0^t U(t, s)y(s)ds)^2$ is minimal, and

2) $U$ can be implemented by a stable LTI system, i.e. the estimate $t \mapsto \int_0^t U(t, s)y(s)ds$ should be the output of a stable LTI system whose input is $y$.

We will call the observers defined above minimax observers.

Motivation The minimax approach is one of many classical ways to pose a state estimation problem. We refer the reader to [12], [4], [14] and [9] for the basic information on the minimax framework. Apart from pure theoretical reasons our interest in the minimax problem is motivated by applications of DAE state estimators in practice. In [23] we briefly discussed one application of DAEs to non-linear filtering problems. Namely, it is well known (see [6]) that the density of a wide class of non-linear diffusion processes solves forward Kolmogorov equation. The latter is a linear parabolic PDE and its analytical solution is usually unavailable. Different approximation techniques exist, though. One can project the density onto a finite dimensional subspace and derive a DAE for the projection coefficients. The resulting DAE will contain additive noise terms which represent the projection error (see [11], [20] for details). The worst-case state estimates of this DAE can be used to construct a state estimate of the non-linear diffusion process.

Besides, DAEs have a wide range of applications, without claiming completeness, we mention robotics [16], cybersecurity [15] and modeling various systems [13]. We conjecture that the results of this paper will be useful for many of the domains in which DAEs are used.

Contribution of the paper In this paper we follow the procedure proposed in [23]: first, we apply a generalization of Kalman’s duality principle in order to transform the minimax estimation problem into a dual optimal control problem for the adjoint DAE. The latter control problem is an infinite horizon linear quadratic optimal control problem for DAEs. Duality allows us to view the observer $U$ as a control input of the adjoint system and to view the worst-case estimation error $\limsup_{t \to \infty} \sup_{f,\eta}(\ell^T Fx(t) - \Omega_U(t))^2$ as the quadratic cost function of the dual control problem. Thus, the solution of the dual control yields an observer whose worst-case asymptotic error is the minimal one. The resulting dual control problem is then solved by translating it to a classical optimal control problem for LTIs. The solution of the latter problem yields a stable autonomous LTI system, whose output is the solution of dual control problem. The translation of the dual control problem to an LTI control problem relies on linear geometric control theory [17], [2]: the state and input trajectories of the DAE correspond to trajectories of an LTI restricted to its largest output zeroing subspace. To sum up, in this paper we solve the (1) minimax estimation problem, and the (2) infinite horizon optimal control problem for DAEs. In addition, we do no impose a-priori restrictions on $F$ and $A$.

Related work To the best of our knowledge, the results of this paper are new. The literature on DAE is vast, but most of the papers concentrate on regular DAEs. The papers [18], [5] are probably the closest to the current paper. However, unlike in [18], we allow non-regular DAEs, and unlike [5], we do not require impulsive observability. In addition, the solution methods are also very different. The finite horizon minimax estimation problem and the corresponding optimal control problem for general DAEs was presented in [23]. A different way of representing solutions of DAEs as outputs of a LTI were presented in [23] too. We note that a feed-back control for finite and infinite-horizon LQ control problems with stationary DAE constraints was constructed.
in [3] assuming that the matrix pencil $F - \lambda A$ was regular. It was mentioned in [23] that transformation of DAE into Weierstrass canonical form may require taking derivative of the model error $f$, which, in turn, leads to restriction of the admissible class of model errors. In contrast, our approach is valid for $L^2$-model errors, which makes it more attractive for applications. Generalized Kalman duality principle for non-stationary DAEs with non-ellipsoidal uncertainty description was introduced in [22] where it was applied to get a sub-optimal infinite-horizon observer. The infinite-horizon LQ control problem for non-regular DAE was also addressed in [19], but unlike this paper, there it is assumed that the DAE has a solution from any initial state. Optimal control of non-linear and time-varying DAEs was also addressed in the literature. Without claiming completeness we mention [8], [7].

**Outline of the paper** This paper is organized as follows. Subsection I-A contains notations, section II describes the mathematical problem statement, section III presents the main results of the paper.

A. Notation

$S > 0$ means $x^T S x > 0$ for all $x \in \mathbb{R}^n$; $F^+$ denotes the pseudoinverse matrix. Let $I$ be either a finite interval $[0, t]$ or the infinite time axis $I = [0, +\infty)$. We will denote by $L^2(I, \mathbb{R}^n)$, $L^2_{loc}(I, \mathbb{R}^n)$ the sets of all square-integrable, and locally square integrable functions $f : I \rightarrow \mathbb{R}^n$ respectively. Recall that a function is locally square integrable, if its restriction to any compact interval is square integrable. If $I$ is a compact interval, then $L^2_{loc}(I, \mathbb{R}^n) = L^2(I, \mathbb{R}^n)$. If $\mathbb{R}^n$ is clear from the context and $I = [0, t]$, $t > 0$, we will use the notation $L^2(0, t)$ and $L^2_{loc}(0, t)$ respectively. If $f$ is a function, and $A$ is a subset of its domain, we denote by $f|A$ the restriction of $f$ to $A$. We denote by $I_n$ the $n \times n$ identity matrix.

II. PROBLEM STATEMENT

Assume that $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$ represent the state vector and output of the following DAE:

$$\begin{align*}
\frac{d(Fx)}{dt} &= Ax(t) + f(t), \\
y(t) &= Hx(t) + \eta(t),
\end{align*}$$

where $F, A \in \mathbb{R}^{n \times n}$, $H \in \mathbb{R}^{p \times n}$, and $f(t) \in \mathbb{R}^n$, $\eta(t) \in \mathbb{R}^p$ stand for the model error and output noise respectively. In this paper we consider the following functional class for DAE’s solutions: if $x$ is a solution on some finite interval $I = [0, t_1]$ or infinite interval $I = (0, +\infty)$, then $x \in L^2_{loc}(I)$, and $Fx$ is absolutely continuous. This allows to consider a state vector $x(t)$ with a non-differentiable part belonging to the null-space of $F$. We refer the reader to [22] for further discussion.

In what follows we assume that for any initial condition $x_0$ and any time interval $I = [0, t_1]$, $t_1 < +\infty$, model error $f$ and output noise $\eta$ are unknown and belong to the given ellipsoidal bounding set $\mathcal{E}(t_1) := \{(x_0, f, \eta) \in \mathbb{R}^n \times L^2(I, \mathbb{R}^n) \times L^2(I, \mathbb{R}^p) : \rho(x_0, f, \eta, t_1) \leq 1\}$, where

$$\rho(x_0, f, \eta, t_1) := x_0^T Q_0 x_0 + \int_0^{t_1} f^T Q f + \eta^T R \eta dt,$$

and $Q_0, Q(t) \in \mathbb{R}^{n \times n}$, $Q_0 = Q_0^T > 0$, $Q = Q^T > 0$, $R \in \mathbb{R}^{p \times p}$, $R^T = R > 0$. In other words, we assume that the triple $(x_0, f, \eta)$ belongs to the unit ball defined by the norm $\rho$.

First, we study the state estimation problem for finite time interval $[0, t_1]$. Our aim is to construct the estimate of the linear function of the state vector $\ell^T Fx(t_1)$, $\ell \in \mathbb{R}^n$, given the output $y(t)$ of (1), $t \in [0, t_1]$. Following [1] we will be looking for an estimate in the class of linear functionals

$$\mathcal{O}_{U, t_1}(y) = \int_0^{t_1} y^T(s) U(s) dt,$$

$U \in L^2(0, t_1)$. Such linear functionals represent linear estimates of a state component $\ell^T Fx(t_1)$ based on past outputs $y$. We will call functions $U \in L^2(0, t_1)$ finite horizon observers. With each such observer $U$ we will associate an observation error defined as follows.

$$\sigma(U, t_1, \ell) := \sup_{(x_0, f, \eta) \in \mathcal{E}(t_1)} (\ell^T Fx(t_1) - \mathcal{O}_{U, t_1}(y))^2.$$

The observation error $\sigma(U, \ell, t_1)$ represents the biggest estimation error of $\ell^T Fx(t_1)$ which can be produced by the observer $U$, if we assume that the initial state and the noise belong to $\mathcal{E}(t_1)$.

So far, we have defined observers which act on finite time intervals. Next, we will define an analogous concept for the whole time axis $[0, +\infty)$.

**Definition 1 (Infinite horizon observers):** Denote by $\mathcal{F}$ the set of all maps $U : \{(t_1, s) \mid t_1 > 0, s \in [0, t_1]\} \rightarrow \mathbb{R}^p$ such that for every $t_1 > 0$, the map $U(t_1, \cdot) : [0, t_1] \rightarrow \mathbb{R}^p$ belongs to $L^2(0, t_1)$.

An element $U \in \mathcal{F}$ will be called an infinite horizon observer. If $y \in L^2_{loc}(I, \mathbb{R}^p)$, $I = [0, t_1]$, $t_1 > 0$ or $I = (0, +\infty)$, then the result of applying $U$ to $y$ is a function $O_U(y) : I \rightarrow \mathbb{R}$ defined by

$$\forall t \in I : O_U(y)(t) = O_{U(t_1, \cdot)}(y) = \int_0^t \ell^T(t, s) y(s) ds.$$

The worst-case error for $U \in \mathcal{F}$ is defined as

$$\sigma(U, \ell) := \limsup_{t_1 \rightarrow \infty} \sigma(U(t_1, \cdot), t_1, \ell).$$

Intuitively, an infinite horizon observer is just a collection of finite horizon observers, one for each time interval. It maps any output defined on some interval (finite or infinite) to an estimate of a component of the corresponding state trajectory. The worst case error of an infinite horizon observer represents the largest asymptotic error of estimating $\ell^T Fx(t)$ as $t \rightarrow \infty$.

The effect of applying an infinite horizon observer $U \in \mathcal{F}$ to an output $y \in L^2_{loc}([0, +\infty), \mathbb{R}^p)$ of the system (1) can
be described as follows. Assume that \( y \) corresponds to some initial state \( x_0 \) and noises \( f \) and \( \eta \) such that
\[
x_0Q_0x_0 + \int_0^{+\infty} f^T(t)Qf(t) + \eta^T(t)R\eta(t)dt \leq 1.
\]
The latter restriction can equivalently be stated as \((x_0, f|_{[0,t_1]}, \eta|_{[0,t_1]}) \in \mathcal{E}(t_1), \forall t_1 > 0\). Assume that \( x \) is the state trajectory corresponding to \( y \). Then \( O_U(y) \) represents an estimate of \( \ell^T Fx \) and the estimation error is bounded from above by \( \sigma(U, \ell) \) in the limit, i.e. for every \( \epsilon > 0 \) there exists \( T > 0 \) such that for all \( t > T \)
\[
\sigma(U, \ell) + \epsilon > (\ell^TFx(t) - O_U(y)(t))^2.
\]
So far we have defined observers as linear maps mapping past outputs to state estimates. For practical purposes it is desirable that the observer is represented by a stable LTI system.

**Definition 2:** The observer \( U \in \mathcal{F} \) can be represented by a stable linear system, if there exists \( A_o \in \mathbb{R}^{r \times r}, B_o \in \mathbb{R}^{r \times p}, C_o \in \mathbb{R}^{1 \times r} \) such that \( A_o \) is stable and for any \( y \in L^2_{loc}(t), I \in [0, t_1], t_1 > 0 \) or \( I = [0, +\infty) \), the estimate \( O_U(y) \) is the output of the LTI system below:
\[
\dot{s}(t) = A_os(t) + B_o \eta(y), \quad s(0) = 0 \quad \forall t \in I : O_U(y)(t) = C_os(t).
\]
The system \( O_U = (A_o, B_o, C_o) \) is called a dynamical observer associated with \( U \).
In addition, we would like to find observers with the smallest possible worst case observation error. These two considerations prompt us to define the minimax observer design problem as follows.

**Problem 1 (Minimax observer design):** Find an observer \( \tilde{U} \in \mathcal{F} \) such that
\[
\sigma(\tilde{U}, \ell) = \inf_{U \in \mathcal{F}} \sigma(U, \ell) < +\infty
\]
and \( \tilde{U} \) can be represented by a stable linear system. In what follows we will refer to such \( \tilde{U} \in \mathcal{F} \) as minimax observer.

### III. MAIN RESULTS

In this section we present our main result: minimax observer for the infinite horizon case. First, in §III-A we present the dual optimal control problem for infinite horizon case. This dual control problem which we are going to formulate is interesting itself. In order to solve the optimal control problem, we will use the concept of output zeroing space from the geometric control. This technique allows us to construct an LTI system whose outputs are solutions of the original DAE. This will be discussed in §III-B. In §III-C we reformulate the dual optimal control problem as a linear quadratic infinite horizon control problem for LTIs. The solution of the latter problem yields a solution to the dual control problem. Finally, in §III-D we present the formulas for the minimax observer and discuss the conditions for its existence.

### A. Dual control problem

We will start with formulating an optimal control problem for DAEs. Later on, we will show that the solution of this control problem yields a solution to the minimax observer design problem. Consider the DAE \( \Sigma \):
\[
dEx = \dot{Ax} + \dot{Bu}(t) \quad \text{and} \quad Ex(0) = E_{x_0}.
\]
Here \( x_0 \in \mathbb{R}^n \) is a fixed initial state and \( \dot{A}, E \in \mathbb{R}^{n \times n}, \dot{B} \in \mathbb{R}^{n \times m} \).

**Notation 1:** Take \( R = \mathbb{R}^{m \times m}, Q, Q_0 \in \mathbb{R}^{n \times n} \) and assume that \( R > 0, Q > 0 \), \( Q_0 \geq 0 \). For any initial state \( x_0 \in \mathbb{R}^n \), and any trajectory \((x, u) \in \mathcal{D}_{x_0}(t), t > t_1 \) define the cost functional
\[
J(x, u, t_1) = x(t_1)^TE_xQ_0Ex(t_1) + \int_0^{t_1} (x^T(s)Qx(s) + u^T(s)Ru(s))ds.
\]
For every \((x, u) \in \mathcal{D}(\infty)\), define
\[
J(x, u) = \limsup_{t_1 \to \infty} J(x, u, t_1).
\]
The infinite horizon optimal control problem for (4) is the problem of finding a tuple of matrices \((A_c, B_c, C_c, C_u)\) such that \( A_c \in \mathbb{R}^{r \times r}, B_c \in \mathbb{R}^{r \times n}, C_c \in \mathbb{R}^{n \times r}, A_c \) is a stable matrix, \( B_cE_{C_c} = I_r \), and for any \( x_0 \in \mathbb{R}^n \) such that \( D_{x_0}(\infty) \neq \emptyset \), the output of the system
\[
\dot{s}(t) = A_cs(t) \quad \text{and} \quad s(0) = B_cEx_0
\]
is such that \((x^*, u^*) \in \mathcal{D}_{x_0}(\infty)\), and
\[
J(x^*, u^*) = \limsup_{t_1 \to \infty} \inf_{(x, u) \in \mathcal{D}_{x_0}(t_1)} J(x, u, t_1)
\]
and the tuple \( \mathcal{C}^* = (A_c, B_c, C_c, C_u) \) will be called the dynamic controller which solves the optimal control problem. For each \( x_0 \), the pair \((x^*, u^*)\) will be called the solution of the optimal control problem for the initial state \( x_0 \).

We will denote infinite horizon control problems above by \( C(E, \dot{A}, B, Q, R, Q_0) \).

Note that the dynamic controller which generates the solutions of the optimal control problem does not depend on the initial condition, in fact, the dynamical controller generates a solution for any initial condition, for which the DAE admits a solution on the whole time axis.

**Remark 1:** The proposed formulation of the infinite horizon control problem is not necessarily the most natural one. We could have also required the \((x^*, u^*) \in \mathcal{D}(\infty)\) to satisfy \( J(x^*, u^*) = \inf_{(x, u) \in \mathcal{D}(\infty)} J(x, u) \). It is easy...
to see that formulation above implies that \( J(x^*, u^*) = \inf_{(x,u) \in \mathcal{D}^*_2} J(x, u) \). Another option could have been to use limit instead of \( \limsup \) in the definition of \( J(x^*, u^*) \) and in (7). In fact, the solution we are going to present remains a solution if we replace \( \limsup \) by limits.

Remark 2 (Solution as feedback): In our case, the optimal control law \( u^* \) can be interpreted as a state feedback. If \( C^* = (A_c, B_c, C_z, C_u) \) is the optimal dynamical controller and \( x_0 \in \mathbb{R}^n \), and \((x^*, u^*)\) is as in (6), then \( s(t) = B_cEC_z s(t) = B_cEx^*(t) \) and thus \( u^*(t) = B_cEx^*(t) \). Note, however, that for DAEs the feedback law does not determine the control input uniquely, since even autonomous DAEs may admit several solutions from the same initial state. If the DAE has at most one solution from any initial state, then the closed-loop system is globally asymptotically stable, i.e. for any initial state the corresponding solution converges to zero.

Now we are ready to present the relationship between Problem 2 and Problem 1.

Definition 3 (Dual control problem): The dual control problem for the observer design problem is the control problem \( \mathcal{C}(F^T, A^T, -H^T, Q^{-1}, R^{-1}, \tilde{Q}_0) \), where
\[
\tilde{Q}_0 = (F^T + z_0 - M_{opt})^TQ^{-1}F^T + z_0 - M_{opt}.
\]
Here \( M_{opt} \) is defined as follows. Let \( r = \text{Rank} F^T \) and \( U \in \mathbb{R}^{n \times (n-r)} \) such that \( \text{im} U = \text{ker} F^T \) and define \( M_{opt} = U(U^TQ_0^{-1}U)^{-1}U^TQ_0^{-1}F^T \).

Theorem 1 (Duality): Let \( \mathcal{D}_1 = (A_1, B_1, C_z, C_u) \) be the dynamical controller solving the dual control problem. Let \((x^*_1, u^*_1)\) be the corresponding solution of the optimal control problem for \( x_0 = \ell \). Then \( \hat{U}(t_1, s) = u^*_1(t_1 - s) \) is the solution of the infinite time horizon observer design problem, and
\[
\sigma(\hat{U}, \ell) = J(x^*, u^*) = \limsup_{t_1 \to \infty} \int_0^{t_1} (u^T(t)R^{-1}u^*(t) + x^*T(t)Q^{-1}x^*(t))dt.
\]
In addition, the dynamical observer \( \mathcal{O}_U \) is of the form
\[
\dot{s}(t) = A^S s(t) + C^Z y(t), s(0) = 0
\]
\[
\mathcal{O}_U(y)(t) = \ell F B^*_F s(t)
\]
Moreover, if \( y \in L^2_{loc}([0, +\infty), \mathbb{R}^p) \) is the output of (1) for \( f = 0 \) and \( \eta = 0 \), then the estimation error \( (\ell F x(t) - \mathcal{O}_U(y(t))) \) converges to zero as \( t \to \infty \).

Note that the matrices of the observer presented in Theorem 1 depend on \( \ell \) only through the equation \( \mathcal{O}_U(y)(t) = \ell F B^*_F s(t) \). Hence, if a solution to the dual control problem exists, then it yields an observer for any \( \ell \), for which the dual DAE
\[
\frac{d(F^T z(t))}{dt} = A^T z(t) - H^T u(t), \quad F^T z(0) = F^T \ell
\]
has a solution defined on the whole time axis.

Theorem 1 implies that existence of a solution of the dual control problem is a sufficient condition for existence of a solution for Problem 1. In fact, we conjecture that this condition is also a necessary one.

Proof: [Proof of Theorem 1] Recall from [23] the following duality principle:

**Proposition 1:** Consider the adjoint DAE:
\[
d(F^T z(t)) = -A^T z(t) + H^T v(t), \quad F^T z(t_1) = F^T \ell.
\]
(1) There exists \( U \in L^2(0, t_1) \) such that \( \sigma(U, \ell, t_1) < +\infty \) iff there exists \( z \in L^2(0, t_1) \) such that \( F^T z \) is absolutely continuous and \( (z, v) \) satisfies (8).

(2) Denote by \( \mathcal{D}_1 \) the set of all tuples \((z, d, v) \in L^2(0, t_1) \times \mathbb{R}^n \times L^2(0, t_1) \) such that \( F^T z \) is absolutely continuous and \((z, v)\) satisfy (8) and \( F^T d = 0 \). For all \((z, d, v) \in \mathcal{D}_1 \), define
\[
\mathcal{I}(z, d, v, t_1) := \int_0^{t_1} (v^T(t)R^{-1}v(t) + z^T(t)Q^{-1}z(t))dt + (F^T F^T z(0) - d)T Q_0^{-1}(F^T F^T z(0) - d)
\]
(9) For any \( U \in L^2(0, t_1) \) such that \( \sigma(U, \ell, t_1) < +\infty \)
\[
\sigma(U, \ell, t_1) = \inf_{z, d, v \in \mathcal{D}_1} \mathcal{I}(z, d, v, t_1).
\]
(3) Moreover, if \( \inf_{U \in L^2(0, t_1)} \sigma(U, \ell, t_1) < +\infty \), then there exists \((z^*, d^*, \hat{U}) \in \mathcal{D}_1 \) such that
\[
\sigma(U, \ell, t_1) = \inf_{(z, d, v) \in \mathcal{D}_1} \mathcal{I}(z, d, v, t_1) = \mathcal{I}(z^*, d^*, \hat{U}, t_1)
\]
(10) Note that in [21] it was proved that the DAE adjoint to (1) has the form (8). Proposition 1 allows us to reduce the problem of minimax observer design to that of finding an optimal controller. To this end, we transform slightly the statement of Proposition 1. First, we get rid of the component \( d \) of the optimization problem from Proposition 1.

**Proposition 2:** Let \((z, v)\) be a solution of (8) such that \( z \in L^2(0, t_1), v \in L^2(0, t_2), F^T z \) is absolutely continuous. Then \( \inf_{d \in \mathbb{R}^n} \mathcal{I}(z, d, v, t_1) = \mathcal{I}(z, \mathcal{M}_{opt}(F^T z(0)), v, t_1) \). Hence, instead of the cost function \( \mathcal{I}(z, d, v, t_1) \), it will be enough to consider the cost function:
\[
\mathcal{F}(z, v, t_1) = \int_0^{t_1} (v^T(t)R^{-1}v(t) + z^T(t)Q^{-1}z(t))dt
\]
Next, we replace the DAE (8) by the DAE of the dual control problem:
\[
\frac{d(F^T x(t))}{dt} = A^T x(t) - H^T u(t) \quad \text{and} \quad F^T x(0) = F^T \ell.
\]
(11) The DAE (11) is obtained from (8) by reversing the time. In order to present the result precisely, we introduce the following notation.

**Notation 2 (\( \delta_{t_1} \)):** If \( r \) is a map defined on \([0, t_1]\), then we denote by \( \delta_{t_1}(r) \) the map \( \delta_{t_1}(r)(t) = r(t_1 - t), t \in [0, t_1] \).
Then \((x, u)\) is a solution of (11) such that \(x \in L^2(0, t_1)\), \(Fx\) is absolutely continuous and \(u \in L^2(0, t_1)\), if and only if \((z, v) = (\delta_{t_1}(x), \delta_{t_1}(u))\) is a solution of (8).

Consider now the dual control problem, and recall that
\[
J(x, u, t_1) = x^T(t_1)FQ_0F^Tx(t_1) + \int_0^{t_1} (u^T(t)R^{-1}u(t) + x^T(t)Q^{-1}x(t))dt.
\]

In addition, recall from Notation 1 that \(\mathcal{D}_l(t_1)\) and \(\mathcal{D}_l(\infty)\) are the sets of solutions \((x, u)\) of (11) defined on the interval \([0, t_1]\) and \([0, +\infty)\) respectively. It is easy to see that
\[
J(x, u, t_1) = \mathcal{J}(\delta_{t_1}(x), \delta_{t_1}(u), t_1).
\]

Hence, Proposition 1 can be reformulated as follows.

**Proposition 3:** There exists \(U \in L^2(0, t_1)\) such that \(\sigma(U, \ell, t_1) < +\infty\) if there exists a solution \((x, u) \in \mathcal{D}(t_1)\) such that \(\delta_{t_1}(u) = U\). If \(U \in L^2(0, t_1)\) is such that \(\sigma(U, \ell, t_1) < +\infty\), then
\[
\sigma(U, \ell, t_1) = \inf_{(x, u) \in \mathcal{D}(t_1), \delta_{t_1}(u) = U} J(x, u, t_1),
\]

There exists a solution \(\hat{U} \in L^2(0, t_1)\) such that \(\sigma(\hat{U}, \ell, t_1) = \inf_{U \in L^2(0, t_1)} \sigma(U, \ell, t_1) < +\infty\), if there exists \((x^*, u^*) \in \mathcal{D}(t_1)\) such that
\[
J(x^*, u^*, t_1) = \inf_{(x, u) \in \mathcal{D}(t_1), \delta_{t_1}(u) = \hat{U}} J(x, u, t_1),
\]

Then \(\hat{U}\) can be chosen as \(\hat{U}(t) = \delta_{t_1}(u^*)\). For all \(x^*, u^*\) the solution of the dual control problem. Since \((x^*, u^*) \in \mathcal{D}(t_1)\) for all \(t_1\), Proposition 3 yields that \(\inf_{U \in L^2(0, t_1)} \sigma(U, \ell, t_1) = \inf_{(x, u) \in \mathcal{D}(t_1)} J(x, u, t_1)\) for all \(t_1 > 0\).

Define \(\hat{U} \in \mathcal{F}\) as \(\hat{U}(t_1, s) = U_{t_1}(s)\) for all \(t_1 > 0, s \in [0, t_1]\). Then it follows that for any \(U \in \mathcal{F}\),
\[
\sigma(\hat{U}, \ell, t_1) = \inf_{U \in \mathcal{F}} \sigma(U, \ell, t_1) \leq \sigma(U(t_1, \cdot), \ell, t_1)\]

and hence \(\sigma(\hat{U}, \ell) = \inf_{U \in \mathcal{F}} \sigma(U, \ell, t_1) < +\infty\). From Proposition 3 it then follows that \(\sigma(\hat{U}^*(\cdot), \ell, t_1) = \inf_{U \in L^2(0, t_1)} \sigma(U, \ell, t_1)\) for all \(t_1 > 0\).

Define now \(\hat{U} \in \mathcal{F}\) as \(\hat{U}(t_1, s) = \delta_{t_1}(u^*), t_1 > 0\). Then \(\sigma(\hat{U}(t_1, \cdot), \ell, t_1) = \inf_{(x, u) \in \mathcal{D}(t_1)} J(x, u^*, t_1)\) and hence
\[
\sigma(\hat{U}, \ell) \leq \sigma(\hat{U}, \ell, t_1) = \limsup_{t_1 \to \infty} \inf_{(x, u) \in \mathcal{D}(t_1)} J(x, u, t_1) = \sigma(\hat{U}, \ell).
\]

and therefore \(\hat{U}\) satisfies (3).

Consider now the dynamical controller \((A_c, B_c, C_u, C_x)\) which is the solution of the dual optimal control problem. Then \(u^*(s) = C_u e^{A_c t}s B_c F^T t\) and thus \(O_t(y)(t_1) = \int_0^{t_1} \hat{U}(t_1, s) y(s)ds = \int_0^{t_1} e^{A_c t}s B_c F^T t e^{A_c (t_1-s)} C_u y(s)ds\). The latter is the output of the linear system \((A^T, C_u^T, \ell F^T B^T)\) for the input \(y\) and the zero initial condition.

Finally, assume that \(y \in L^2_{loc}([0, +\infty), \mathbb{R}^p)\) is the output of the DAE (1) for the state trajectory \(x\) and \(f = 0\) and \(n = 0\). Let \((x^*, u^*)\) be the solution to the dual control problem. Consider the derivative of \(r(t) = x^T(t) F^T x^*(t_1) - x^T(t) F^T x^*(t_1-t) + x^T(t) F^T (F^T t e^{A_c (t_1-t)} - x^T(t) A^T x^*(t_1-t)) + + x^T(t) Hu^*(t_1-t) = u^T (t_1-t) y(t)\)

and hence
\[
O_t(y)(t_1) = \int_0^{t_1} \hat{r}(s)ds = x^T(0) F^T x^*(t_1) - x^T(t_1) F^T x^*(0).
\]

By noticing that \(x^*(0) = 0\), it follows that
\[
(e^{A_c t_1} - O_t(y)(t_1)) = x^T(0) F^T x^*(t_1).
\]

Since \(F^T x^*(t_1)\) converges to zero as \(t_1 \to \infty\), then the estimation error will also converge to zero.

**B. DAE systems as solutions to the output zeroing problem**

Consider the DAE system (4). In this section we will study solution set \(\mathcal{D}_{x_0}(t_1)\) in \([0, +\infty)\) of (4). It is well known that for any fixed \(x_0\) and \(u, (4)\) may have several solutions or no solution at all. In the sequel, we will use the tools of geometric control theory to find a subset \(X \subseteq \mathbb{R}^n\), such that for any \(x_0 \in E^{-1}(X)\), \(\mathcal{D}_{x_0}(t_1) \neq \emptyset\) for all \(t_1 \in [0, +\infty)\). Furthermore, we provide a complete characterization of all such solutions as outputs of an LTI system.

**Theorem 2:** Consider the DAE system (4). There exists a linear system \(\mathcal{J} = (A_1, B_1, C_1, D_1)\) with \(A_1 \in \mathbb{R}^{n \times n}, B_1 \in \mathbb{R}^{n \times k}, C_1 \in \mathbb{R}^{(n+m) \times n}\) and \(D_1 \in \mathbb{R}^{(n+m) \times k}\), \(n \leq n\), and a linear subspace \(X \subseteq \mathbb{R}^n\) such that the following holds.

- \(\text{Rank} D_1 = k\).
- Consider the partitioning \(C_1 = [C_1^T, C_1_{inp}^T]\), \(D_1 = [D_1^T, D_{1_{inp}}^T]\), \(C_1 \in \mathbb{R}^{n \times n}, C_1_{inp} \in \mathbb{R}^{m \times n}, D_1 \in \mathbb{R}^{n \times k}, D_{1_{inp}} \in \mathbb{R}^{m \times k}\). Then \(ED_1 = 0\), \(\text{Rank} EC_1 = n\), \(X = \text{im} EC_1\).
- For any \(t_1 \in [0, +\infty)\),
\[
\mathcal{D}_{x_0}(t_1) \neq \emptyset \iff E x_0 \in X.
\]

Define the map \(M = (EC_x)^+ : X \to \mathbb{R}^\hat{n}\). Then \((x, u) \in \mathcal{D}_{x_0}(t_1)\) for some \(t_1 \in [0, +\infty)\) if and only if there exists some input \(g \in L^2(I, \mathbb{R}^k)\), \(I = [0, t_1] \cap [0, +\infty)\), such that
\[
\dot{x} = A_1 v + B_1 g \quad \text{and} \quad v(0) = M(E x_0),
\]
\[
x = C_1 v + D_{1_{inp}} g,
\]
\[
u = C_1_{inp} + D_{1_{inp}} g.
\]

Moreover, in this case, the state trajectories \(x\) and \(u\) are related as \(M(E x) = \nu\).
Proof: [Proof of Theorem 2] There exist suitable non-singular matrices $S$ and $T$ such that

$$SET = \begin{bmatrix} I_\ell & 0 \\ 0 & 0 \end{bmatrix},$$

where $r = \text{Rank}E$. Let

$$S \tilde{A} T = \begin{bmatrix} \tilde{A} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad S \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

be the decomposition of $A, B$ such that $A_{11} \in \mathbb{R}^{r \times r}, B_{11} \in \mathbb{R}^{r \times m}$. Define

$$G = \begin{bmatrix} A_{12} \\ B_1 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} A_{22} \\ B_2 \end{bmatrix} \quad \text{and} \quad \bar{C} = A_{21}.$$ 

Consider the following linear system

$$S \begin{cases} \dot{p} = \bar{A} p + G q \\ z = \bar{C} p + \tilde{D} q \end{cases}.$$

(13)

The trajectories $(x, u)$ of the DAE (4) are exactly those trajectories $(p, q), T^{-1} x = (p^T, q^T)^T, q = (q_1^T, u^T)^T, q_1 \in \mathbb{R}^{n-r},$ of the linear system (13) for which the output $z$ is zero.

Recall from [17, Section 7.3] the problem of making the output zero by choosing a suitable input. Recall from [17, Definition 7.6] the concept of a weakly observable subspace of a linear system. If we apply this concept to $S$, then an initial state $p(0) \in \mathbb{R}^r$ of $\bar{S}$ is weakly observable, if there exists an input function $q \in L^2([0, +\infty), \mathbb{R}^k)$ such that the resulting output function $z$ of $S(\Sigma)$ equal zero, i.e. $z(t) = 0$ for all $t \in [0, +\infty)$. Following the convention of [17], let us denote the set of all weakly observable initial states by $\mathcal{V}(S)$. As it was remarked in [17, Section 7.3], $\mathcal{V}(S)$ is a vector space and in fact it can be computed. Moreover, if $p(0) \in \mathcal{V}(S)$ and for the particular choice of $q, z = 0$, then $p(t) \in \mathcal{V}(S)$ for all $t \geq 0$.

Let $I = [0, t]$ or $I = [0, +\infty)$. Let $q \in L^2(I, \mathbb{R}^{n-r+m})$ and let $p_0 \in \mathbb{R}^r$. Denote by $p(p_0, q)$ and $z(p_0, q)$ the state and output trajectory of (13) which corresponds to the initial state $p_0$ and input $q$. For technical purposes we will need the following easy extension of [17, Theorem 7.10–11].

Theorem 3: 1) $\mathcal{V} = \mathcal{V}(S)$ is the largest subspace of $\mathbb{R}^r$ for which there exists a linear map $\bar{F} : \mathbb{R}^r \to \mathbb{R}^{m+n-r}$ such that

$$\left(\bar{A} + G \hat{F}\right)\mathcal{V} \subseteq \mathcal{V} \quad \text{and} \quad (\bar{C} + D \hat{F})\mathcal{V} \subseteq \mathcal{V}$$

(14)

2) Let $\bar{F}$ be a map such that (14) holds for $\mathcal{V} = \mathcal{V}(S)$. Let $L \in \mathbb{R}^{(n+r-u-r) \times k}$ for some $k$ be a matrix such that $\text{im} L = \ker \tilde{D} \cap G^{-1}(\mathcal{V}(S))$ and $\text{Rank} L = k$.

For any interval $I = [0, t]$ or $I = [0, +\infty)$, and for any $p_0 \in \mathbb{R}^r, q \in L^2_{loc}(I, \mathbb{R}^k)$,

$$z(p_0, q)(t) = 0 \quad \text{for} \quad t \in I \quad \text{a.e.}$$

if and only if $p_0 \in \mathcal{V}$ and there exists $w \in L^2_{loc}(I, \mathbb{R}^{n-r+m})$ such that

$$q(t) = \hat{F} p_0(q)(t) + Lw(t) \quad \text{for} \quad t \in I \quad \text{a.e.}$$

We are ready now to finalize the proof of Theorem 2. The desired linear system $\mathcal{X} = (A_1, B_1, C_1, D_1)$ is now obtained as follows. Consider the linear system below.

$$\dot{p} = (\tilde{A} + G \hat{F}) p + GLw$$

$$\left(\begin{array}{c} x^T \\ u^T \end{array}\right) = \bar{C} p + \tilde{D} w$$

Choose a basis of $V = \mathcal{V}(S)$ and choose $(A_1, B_1, C_1, D_1)$ as follows: $D_1 = \bar{D}$, and let $A_1, B_1, C_1$ be the matrix representations in this basis of the linear maps $(A + G F) : V \to V, GL : \mathbb{R}^k \to V,$ and $\bar{C} : V \to \mathbb{R}^{n+m}$ respectively. Define

$$\mathcal{X} = \left\{ S^{-1} \begin{bmatrix} p \\ 0 \end{bmatrix} \mid p \in \mathcal{V} \right\}.$$ 

It is easy to see that this choice of $(A_1, B_1, C_1, D_1)$ and $\mathcal{X}$ satisfies the conditions of the theorem.

Remark 4 (Regular case): The well-known case when (4) is regular, i.e. when $\det(s E - A) \neq 0$ has the following interpretation. In this case the linear system $\mathcal{S}$ from the proof of Theorem 2 is left invertible, and $\mathcal{V}(S) = \mathbb{R}^r$. The proof of Theorem 2 is constructive and yields an algorithm for computing $(A_1, B_1, C_1, D_1)$ from $(E, A, B)$. This prompts us to introduce the following terminology.

Definition 4: A linear system $\mathcal{X} = (A_1, B_1, C_1, D_1)$ described in the proof of Theorem 2 is called the linear system associated with the DAE (4).

Note that the linear system associated with $(E, \tilde{A}, \tilde{B})$ is not unique. There are two sources of non-uniqueness:

1) The choice of the matrices $S$ and $T$ in (12).
2) The choice of $F$ and $L$ in Theorem 3.

However, we can show that all associated linear systems are feedback equivalent.

Definition 5 (Feedback equivalence): Two linear systems $\mathcal{X}_i = (A_i, B_i, C_i, D_i), i = 1, 2$ and are said to be feedback equivalent, if there exist a linear state feedback matrix $K$ and a non-singular square matrix $U$ such that $(A_1 + B_1 K, B_1 U, C_1 + D_1 K, D_1 U)$ and $\mathcal{X}_2$ are algebraically similar.

Lemma 1: Let $\mathcal{X}_i = (A_i, B_i, C_i, D_i), i = 1, 2$ be two linear systems which are obtained from the proof of Theorem 2. Then $\mathcal{X}_1$ and $\mathcal{X}_2$ are feedback equivalent.

The proof of Lemma 1 can be found in the appendix.

C. Solution of the optimal control problem for DAE

We apply Theorem 2 in order to solve a control problem defined in Problem 2. Let $\mathcal{X} = (A_1, B_1, C_1, D_1)$ be a linear system associated with $\Sigma$ and let $M$ be the map described in Theorem 2 and let $C_i$ the component of $C_1$ as defined in Theorem 2. Consider the following linear quadratic control problem. For every initial state $v_0$, for every interval $I$ containing $[0, t_1]$ and for every $g \in L^2_{loc}(I, \mathbb{R}^k)$ define the
cost functional \( J(v_0, g, t) \)
\[
J(v_0, g, t) = v^T(t_1) E^T C_s^T Q_0 E C_s v(t_1) + \\
+ \int_{t_0}^{t_1} \nu^T(t) \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \nu(t) dt
\]
\[
\dot{v} = A v + B_2 g \\
\nu = C v + D_1 g.
\]

For any \( g \in L^2_{\text{loc}}([0, t_1], \mathbb{R}^k) \) and \( v_0 \in \mathbb{R}^n \), define
\[
\mathcal{J}(v_0, g) = \limsup_{t_1 \to +\infty} \mathcal{J}(v_0, g, t_1).
\]

Consider the control problem of finding for every initial state \( v_0 \) an input \( g^* \in L^2_{\text{loc}}(\mathbb{R}^k) \) such that
\[
\mathcal{J}(v_0, g^*) = \limsup_{t_1 \to +\infty} \inf_{g \in L^2_{\text{loc}}(0, t_1)} \mathcal{J}(v_0, g, t_1).
\]

**Definition 6 (Associated LQ problem):** The control problem (15) is called an LQ problem associated with \( C(E, A, B, Q, R, Q_0) \) and it is denoted by \( \mathcal{L}(A_1, B_1, C_1, D_1) \).

**Remark 5 (Uniqueness):** Note the solution of an associated LQ does not depend on the choice of \( \mathcal{J} \): for any two choices of \( \mathcal{J} \), the corresponding solutions can be transformed into each other by a linear state feedback and linear coordinate changes of the input- and state-space.

The relationship between the associated LQ problem and the original control problem for DAEs is as follows.

**Theorem 4:** Let \( g^* \in L^2_{\text{loc}}([0, t_1], \mathbb{R}^k) \) and let \((x^*, u^*)\) be the corresponding output of \( \mathcal{J} = (A_1, B_1, C_1, D_1) \) from the initial state \( v_0 = M(E_x_0) \) for some \( x_0 \in \mathbb{R}^n \), \( \mathcal{J}_0(\infty) \neq 0 \). Then \((x^*, u^*) \in \mathcal{J}_0(\infty) \) and \( g^* \) is a solution of \( \mathcal{L}(A_1, B_1, C_1, D_1) \) for \( v_0 \) if and only if
\[
J(x^*, u^*) = \limsup_{t_1 \to +\infty} \inf_{(x, u) \in \mathcal{J}_0(t_1)} J(x, u, t).
\]

**Proof:** [Proof of Theorem 5] Let \( \mathcal{J} = (A_1, B_1, C_1, D_1) \) be the LQ problem associated with \( C(E, A, B, Q, R, Q_0) \). Assume that \( (A_1, B_1) \) is stabilizable. Define \( S = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \). Consider the algebraic Riccati equation
\[
0 = PA_1 + A_1^T P - K^T (D_1^T S D_1) K + C_1^T S C_1,
\]
\[
K = (D_1^T S D_1)^{-1} (B_1^T P + D_1^T S C_1).
\]

Then (16) has a unique solution \( P > 0 \), and \( A_1 - B_1 K \) is a stable matrix. Moreover, if \( g^* \) is defined as
\[
\dot{v}^* = A_1 v^* + B_1 g^* \quad \text{and} \quad v^*(0) = v_0
\]
\[
g^* = -K v^*,
\]
then \( g^* \) is a solution of \( \mathcal{L}(A_1, B_1, C_1, D_1) \) for the initial state \( v_0 \) and \( v_0^T P v_0 = \mathcal{J}(v_0, g^*) \).

**Proof:** [Proof of Theorem 5] Let us first apply the feedback transformation \( g = F v + U w \) to \( \mathcal{J} = (A_1, B_1, C_1, D_1) \) with \( U = -(D_1^T S D_1)^{-1/2} \) and \( F = -(D_1^T S D_1)^{-1} D_1^T S C_1 \). Consider the linear system
\[
\dot{v} = (A_1 + B_1 F) v + B_1 U w \quad \text{and} \quad v(0) = v_0
\]

For any \( w \in L^2_{\text{loc}}(I) \), where \( I = [0, t_1] \) of \( I = [0, +\infty) \), the state trajectory \( v \) of (18) equals the state trajectory of \( \mathcal{J} \) for the input \( g = F v + U w \) and initial state \( v_0 \). Moreover, all inputs \( g \) of \( \mathcal{J} \) can be represented in such a way. Define now
\[
\mathcal{J}(v_0, w, t) = v^T(t) E^T C_s^T Q_0 E C_s v(t) + \\
+ \int_0^t (v^T(t) (C_1 + D_1 F)^T S(C_1 + D_1 F) v(t) + w^T(t) w(t)) dt,
\]
where \( w \) is a solution of (18). It is easy to see that for \( g = F v + U w \), \( \mathcal{J}(v_0, g, t) = \mathcal{J}(v_0, w, t) \).

Consider the problem of minimizing \( \lim_{t_1 \to +\infty} \mathcal{J}(v_0, w, t) \). The solution of this problem can be found using [10, Theorem 3.7]. To this end, notice that \( (A_1 + B_1 F, B_1 U) \) is stabilizable and \( (S^{1/2} (C_1 + D_1 F), A_1 + B_1 F) \) is observable. Indeed, it is easy to see that stabilizability of \( (A_1, B_1) \) implies that \( (A_1 + B_1 F, B_1 U) \). Observability \( (S^{1/2} (C_1 + D_1 F), A_1 + B_1 F) \) is implied by the fact that by Theorem 2, \( E C_s \) is full column rank and \( ED_s = 0 \), and thus \( E (C_1 + D_1 F) = E C_s \) is full column rank. Furthermore, notice that (16) is equivalent to the algebraic Riccati equation described in [10, Theorem 3.7] for the problem of minimizing \( \lim_{t_1 \to +\infty} \mathcal{J}(v_0, w, t) \). Hence, by [10, Theorem 3.7], (16) has a unique positive definite solution \( P \), and \( A_1 - B_1 (F + U^T B_1 P) = A_1 - B_1 K \) is a stable matrix. From [10, Theorem 3.7], there exists \( w^* \) such that \( \lim_{t_1 \to +\infty} \mathcal{J}(v_0, w^*, t) \) is minimal, and \( v_0^T P v_0 = \lim_{t_1 \to +\infty} \mathcal{J}(v_0, w^*, t) \) from [10, Theorem 3.7] we can also deduce that \( v_0^T P v_0 = \lim_{t_1 \to +\infty} \inf_{w \in L^2_{\text{loc}}(0, t_1)} \mathcal{J}(v_0, w, t_1) \).

Hence, \( g^* = F v^* + U w^* \) is a solution of \( \mathcal{L}(A_1, B_1, C_1, D_1) \) for the initial state \( v_0 \), where \( v^* \) is the solution of (18) which corresponds to \( w = w^* \). A routine computation reveals that \( (v^*, g^*) \) satisfies (17).

**Corollary 1:** Consider the control problem \( C(E, A, B, Q, R, Q_0) \) and let \( \mathcal{L}(A_1, B_1, C_1, D_1) \) be an LQ problem associated with \( C(E, A, B, Q, R, Q_0) \). Assume that \( (A_1, B_1) \) is stabilizable. Let \( P \) be the unique positive definite solution of (16) and let \( K \) be as in (16). Let \( C_s, C_{in}, D_s, D_{in} \) be the decomposition of \( C_1 \) and \( D_1 \) as defined in Theorem 2 and let \( M = (E C_s)^+ \). Then the dynamical controller \( \hat{v} = (A_c, B_c, C_x, D_c) \) with
\[
A_c = A_1 - B_1 K, \quad C_x = C_s - D_s K, \quad C_u = (C_{in} - D_{in} K) \quad \text{and} \quad B_c = M,
\]
then \( g^* \) is a solution of \( C(E, A, B, Q, R, Q_0) \).
Remark 6 (Computation and existence of a solution):
The existence of solution for Problem 2 and its computation depend only on the matrices \((E, A, B, Q, R, Q_0)\). Indeed, a linear system \(\mathcal{S}\) associated with \((E, A, B)\) can be computed from \((E, \hat{A}, \hat{B})\), and the solution of the associated LQ problem can be computed using \(\mathcal{S}\) and the matrices \(Q, Q_0, R\). Notice that the only condition for the existence of a solution is that \(\mathcal{S} = (A_l, B_l, C_l, D_l)\) is stabilizable. Since all linear systems associated with the given DAE are feedback equivalent, stabilizability of an associated linear system does not depend on the choice of the linear system. Thus, stabilizability of \(\mathcal{S}\) can be regarded as a property of \((E, \hat{A}, \hat{B})\). The link between stabilizability of \(\mathcal{S}\) and the classical stabilizability for DAEs remains a topic for future research.

D. Observer design for DAE

By applying Corollary 1 and Theorem 1, we obtain the following procedure for solving Problem 1.

- **Step 2.** Check if \((A_l, B_l)\) is stabilizable. If it is, let
  \[
  X = \begin{bmatrix}
  Q^{-1} & 0 \\
  0 & R^{-1}
  \end{bmatrix}.
  \]
  Consider the algebraic Riccati equation
  \[
  0 = PA_l + A_l^T P - K^T (D_l^T X D_l) K + C_l^T X C_l,
  \]
  \[
  K = (D_l^T X D_l)^{-1} (B_l^T P + D_l^T X C_l).
  \]
  The equation (19) has a unique solution \(P > 0\).

- **Step 3.** The dynamical observer \(\hat{O}_d\) which is a solution of Problem 1 is of the form:
  \[
  \dot{r}(t) = (A_l - B_l K) r(t) + (C_l - D_l K) T \begin{bmatrix} 0 \\ y(t) \end{bmatrix}
  \]
  \[
  \hat{O}_d(y)(t) = \ell^T F M T r(t),
  \]
  and
  \[
  \hat{U}(t, s) = (C_l - D_l K)^{e(A_l - B_l K)(t-s)}, M F T \ell.
  \]
  The observation error equals
  \[
  \sigma(\hat{U}, \ell) = \ell^T F M T P M F T \ell.
  \]
  Recall that \(M = (F^T C_s)^+\), where \(C_s\) is the submatrix of \(C_l\) formed by its first \(n\) rows.

Remark 7 (Conditions for existence of an observer):
The existence of the observer above depends only on whether the chosen linear system associated with the dual DAE is stabilizable. As it was mentioned before, the latter is a property of the tuple \((F, A, H)\). Hence, the property that the linear system associated with the dual DAE is stabilizable could be thought of as a sort of detectability property. The relationship between this property and the detectability notions established in the literature remains a topic of future research.

IV. CONCLUSIONS

We have presented a solution to the minimax observer design problem and the infinite horizon linear quadratic control problem for linear DAEs. We have also shown that these two problems are each other’s dual. The main novelty of this contribution is that we made no solvability assumptions on DAEs. The only condition we need is that the LTI associated with the dual DAE should be stabilizable. We conjecture that this condition is also a necessary one. The clarification of this issue remains a topic of future research.

REFERENCES

APPENDIX

Proof: [Proof of Lemma 1] We will use the following terminology in the sequel. Consider two linear systems $(A_1, B_1, C_1, D_1)$ and $(A_2, B_2, C_2, D_2)$ with $n$ states, $p$ outputs and $m$ inputs. A tuple $(T, F, G, U)$ of matrices, $T \in \mathbb{R}^{n \times n}$, $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{p \times p}$, $F \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{n \times p}$ such that $T, U$ and $V$ are non-singular, is said to be a feedback equivalence with output injection from $(A_1, B_1, C_1, D_1)$ to $(A_2, B_2, C_2, D_2)$, if

$$T(A_1 + B_1 F + G C_1 + G D_1 F) T^{-1} = A_2$$

$$V(C_1 + D_1 F) V^{-1} = C_2$$

$$T(B_1 + G D_1) U = B_2$$

If $G = 0$, $V = I_p$, then $(T, F, G, U)$ is just a feedback equivalence and $(A_1, B_1, C_1, D_1)$ and $(A_2, B_2, C_2, D_2)$ are feedback equivalent. In this case (i.e. when $G = 0$, $V = I_p$), we denote this transformation by $(T, F, U)$.

Let $S_i, T_i \in \mathbb{R}^{n \times n}$ be invertable, such that $S_i ET_i = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, $i = 1, 2$. Let

$$S_i \hat{A} T_i = \begin{bmatrix} A_i & A_{12,i} \\ A_{21,i} & A_{22,i} \end{bmatrix}$$

$$S_i \hat{B} = \begin{bmatrix} B_{1, i} \\ B_{2, i} \end{bmatrix}$$

$$G_i = \begin{bmatrix} A_{12, i} & B_{1, i} \end{bmatrix}$$

$$\bar{C}_i = A_{21,i}$$

$$\bar{D}_i = [A_{22,i}, B_{2,i}]$$

and consider the linear systems

$$\mathcal{A}_i \begin{bmatrix} \dot{p}_i \\ z_i \\ \bar{p}_i \end{bmatrix} = \begin{bmatrix} A_i p_i + G_i q_i \\ \bar{C}_i p_i + \bar{D}_i q_i \end{bmatrix}$$

for $i = 1, 2$. Denote by $\mathcal{V}_i = \mathcal{V}(\mathcal{A}_i)$ the set of weakly observable states of $\mathcal{A}_i, i = 1, 2$. Denote by $\mathcal{F}(\mathcal{V}_i)$, $i = 1, 2$, the set of all state feedback matrices $F \in \mathbb{R}^{n \times m}$ such that $(A_i + G_i F) \mathcal{V}_i \subseteq \mathcal{V}_i$, $(C_i + D_i F) \mathcal{V}_i = 0$. Pick $F_i \in \mathcal{F}(\mathcal{V}_i)$, $i = 1, 2$ and pick full column rank matrices $L_i$, $i = 1, 2$ such that $\text{im} L_i = \mathcal{G}_i^{-1}(\mathcal{V}_i) \cap \ker D_i$. In order to prove the lemma, it is enough to show that $\text{rank} L_1 = \text{rank} L_2 = k$, and there exist invertable linear maps $T \in \mathbb{R}^{r \times r}$, $U \in \mathbb{R}^{k \times k}$, and a matrix $F \in \mathbb{R}^{r \times k}$ such that

$$T(V_1) = V_2$$

$$(A_1 + G_1 F_1 + G_1 L_1 F)V_1 \subseteq V_1$$

$$\forall x \in \mathcal{V}_1 :$$

$$T(A_1 + G_1 F_1 + G_1 L_1 F) T^{-1} x = (A_2 + G_2 F_2) x$$

$$T G_1 L_1 U = G_2 L_2$$

$$[T_1 \ 0 \ I_m \ L_1 U] [0_{r \times k} \ 0_{r \times k}] = [T_2 \ 0 \ I_m \ L_2]$$

$$T_1 \ 0 \ I_m \ (F_1 + L_1 F) x = T_2 \ 0 \ I_m \ F_2 T x,$$

where $0_{r \times k}$ denotes the $r \times k$ matrix with all zero entries. Indeed, the associated linear systems arising from the two choices $T_1, S_i, F_i, L_i$, $i = 1, 2$ are in fact isomorphic to the following linear system defined on $\mathcal{V}_i$, $i = 1, 2$,

$$\mathcal{L}_i \begin{bmatrix} \dot{p}_i \\ (x, u) \end{bmatrix} T = \begin{bmatrix} T_i & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} F_i | V, p + G_i L_i w \\ u \end{bmatrix}$$

(21)

If $(T, F, U)$ satisfy (20), it then follows that $(T, F, U)$ is a feedback equivalence between $L_1$ and $L_2$.

In order to find the matrices $F, U, T$, notice that

$$T_2^{-1} T_1 = \begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix}$$

for $R_{11} \in \mathbb{R}^{r \times r}$, $R_{22} \in \mathbb{R}^{L_{1, r} \times (n-r)}$, $R_{21} \in \mathbb{R}^{L_{1, r} \times (r-r)}$.

Indeed, assume $T_2^{-1} T_1 \begin{bmatrix} 0 \\ q \end{bmatrix} = \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix}$ for some $q, \bar{q} \in \mathbb{R}^{n-r}$, $\bar{p} \in \mathbb{R}^r$. Then

$$\begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix} = S_2 E T_2 T_2^{-1} T_1 \begin{bmatrix} 0 \\ q \end{bmatrix} = S_2 S_1^{-1} S_1 E T_1 \begin{bmatrix} 0 \\ q \end{bmatrix} = S_2 S_1^{-1} 0 = 0.$$

Hence, $T_2^{-1} T_1 \begin{bmatrix} 0 \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{q} \end{bmatrix}$ from which the statement follows.

In a similar fashion

$$S_2 S_1^{-1} = \begin{bmatrix} H_{11} & H_{21} \\ 0 & H_{22} \end{bmatrix},$$

where $H_{11} \in \mathbb{R}^{r \times r}$, $H_{22} \in \mathbb{R}^{(n-r) \times (n-r)}$, $H_{12} \in \mathbb{R}^{r \times (n-r)}$, moreover,

$$H_{11} = R_{11}.$$

Indeed,

$$S_2 S_1^{-1} \begin{bmatrix} p \end{bmatrix} = S_2 S_1^{-1} S_1 E T_1 \begin{bmatrix} p \end{bmatrix} =$$

$$S_2 E T_2 T_2^{-1} T_1 \begin{bmatrix} p \end{bmatrix} = \begin{bmatrix} \bar{p} \\ 0 \end{bmatrix}$$

for some $\bar{p} \in \mathbb{R}^r$. Finally,

$$\begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix} = S_2 S_1^{-1} (S_1 E T_1) = (S_2 E T_2) T_2^{-1} T_1 = \begin{bmatrix} R_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $R_{11} = H_{11}$.

From $S_2 \hat{A} T_2 = S_2 S_1^{-1} S_1 \hat{A} T_1 (T_2^{-1} T_1)^{-1}$ it and $S_2 \hat{B} = S_2 S_1^{-1} S_1 \hat{B}$ follows that

$$A_2 = R_{11}(A_1 + G_1 F + \hat{G} \bar{C}_1 + \hat{G} \bar{D}_1 \hat{F}) R_{11}^{-1}$$

$$G_2 = R_{11}(G_1 + \hat{G} \bar{D}_1 \hat{F})$$

$$\hat{D}_2 = \hat{V} \bar{D}_1 \hat{U}$$

$$\hat{C}_2 = \hat{V} (\bar{C}_1 + \hat{D}_1 \hat{F}) R_{11}^{-1}$$

where $\hat{F} = \begin{bmatrix} -R_{22}^{-1} R_{12} \\ 0 \end{bmatrix}$, $G = R_{11}^{-1} H_{12}$, $\hat{U} = \begin{bmatrix} R_{22}^{-1} & 0 \\ 0 & I_m \end{bmatrix}$, and $\hat{V} = H_{22}$.

We then claim that the following choice of matrices

$$T = R_{11}$$

$$U = L_1^+ \hat{U} L_2$$

$$F = L_1^+ (\hat{F} + \hat{U} F R_{11} - F_1)$$

(23)

satisfies (20). We prove (20a) – (20f) one by one.
Proof of (20a): Indeed, from (22) it then follows that $\mathcal{X}_1$ and $\mathcal{X}_2$ are related by a feedback equivalence with output injection $(R_{11}, F, G, U, V)$. From [17, page 169, Exercise 7.1] it follows that $V_2 = R_{11}(V_1) = TV_1$.

Proof of (20b): From the definition of $F_2$ it follows $(C_2 + D_2 F_2) V_2 = \{0\}$, and $(A_2 + G_2 F_2) V_2 \subseteq V_2$. Substituting the expressions for $C_2, D_2, A_2, G_2$ from (22) and using that $V_2 = R_{11} V_1$ and that $R_{11}, V$ are invertible, it follows that for all $x \in V_1$,

$$
(\hat{C}_1 + \hat{D}_1 (\hat{F} + \hat{U} F R_{11})) x = 0
$$

and

$$(A_1 + G_1 (\hat{F} + \hat{U} F R_{11}) + \hat{G} \hat{C}_1 + \hat{G} D_1 (\hat{F} + \hat{U} F R_{11})) x = 0$$

for all $x \in V_1$.

Hence, $(A_1 + G_1 F_1 + G_1 F) V_1 \subseteq V_1$.

Proof of (20c): Since from the definition of $F_1$ it follows that $(A_1 + G_1 F_1) x \in V_1$, $(C_1 x + D_1 F_1 x) = 0$, for all $x \in V_1$, from (24), it then follows that for all $x \in V_1$,

$$(A_1 + G_1 (\hat{F} + \hat{U} F R_{11}) x = \hat{G} \hat{C}_1 + \hat{G} D_1 (\hat{F} + \hat{U} F R_{11}) x = 0$$

Hence, $(\hat{F} + \hat{U} F R_{11} - F_1) x \in \ker L_1$ and hence

$$L_1 F x = L_1 \hat{F} R_{11} x = (\hat{F} + \hat{U} F R_{11} - F_1) x$$

for all $x \in V_1$. From this it follows that

$$x \in V_1 : F_1 x + L_1 F x = (\hat{F} + \hat{U} F R_{11}) x. \quad (25)$$

From (25) it then follows that $(A_1 + G_1 F_1 + G_1 L_1 F) x = A_1 x + G_1 (\hat{F} + \hat{U} F R_{11}) x$ for all $x \in V_1$. From this and (22), (20c) follows.

Proof of (20d): Recall that $\ker L_2 = \ker (\hat{V} D_1 \hat{U}) \cap (R_{11} G_1 \hat{U})^{-1} (V_2) = \hat{U}^{-1} (\ker D_1 \cap G_1 (V_1)) = \hat{U}^{-1} \ker L_1$.

Since $\hat{U}$ is invertible, it follows that $\text{Rank} L_1 = \text{Rank} L_2 = k$ and that

$$L_1 U = L_1 L_1^+ \hat{U} L_2 = \hat{U} L_2. \quad (26)$$

Hence, using (22) and $\hat{D}_2 \hat{U} L_2 = 0$, it follows that $T G_1 L_1 U = T G_1 U L_2 = T G_1 L_2 + T G_1 L_2 = G_2 L_2$.

Proof of (20e): It is easy to see that (20e) is equivalent to

$$
\begin{bmatrix}
T_2 & 0 \\
0 & I_m
\end{bmatrix}
^{-1}
\begin{bmatrix}
T_1 & 0 \\
0 & I_m
\end{bmatrix}
\begin{bmatrix}
0_{r \times k} \\
L_1 U
\end{bmatrix} = \begin{bmatrix}
0_{r \times k} \\
L_2
\end{bmatrix}. \quad (27)
$$

We will show (27) To this end, notice that

$$
\begin{bmatrix}
T_2 & 0 \\
0 & I_m
\end{bmatrix}
^{-1}
\begin{bmatrix}
T_1 & 0 \\
0 & I_m
\end{bmatrix} = \begin{bmatrix}
T_2^{-1} T_1 & 0 \\
0 & I_m
\end{bmatrix} = \begin{bmatrix}
R_{11} & 0 \\
R_{21} & R_{22}
\end{bmatrix}
\begin{bmatrix}
0_{r \times k} \\
L_1 U
\end{bmatrix} = \begin{bmatrix}
0_{r \times k} \\
\hat{U}^{-1} L_1 U
\end{bmatrix}\quad \text{(28)}
$$

Hence,

$$
\begin{bmatrix}
T_2 & 0 \\
0 & I_m
\end{bmatrix}
^{-1}
\begin{bmatrix}
T_1 & 0 \\
0 & I_m
\end{bmatrix}
\begin{bmatrix}
0_{r \times k} \\
L_1 U
\end{bmatrix} = \begin{bmatrix}
0_{r \times k} \\
\hat{U}^{-1} L_1 U
\end{bmatrix}. \quad (29)
$$

Using $L_1 U = \hat{U} L_2$ proven above in (26), it follows that

$$\hat{U}^{-1} L_1 U = L_2$$

and hence (29) implies (27).

Proof of (20f): Again, it is enough to show that

$$\forall x \in V_1 : \begin{bmatrix}
T_2 & 0 \\
0 & I_m
\end{bmatrix}
^{-1}
\begin{bmatrix}
T_1 & 0 \\
0 & I_m
\end{bmatrix}
\begin{bmatrix}
I_r \\
F_1 + L_1 F
\end{bmatrix} x = \begin{bmatrix}
I_r \\
F_2
\end{bmatrix} R_{11} x. \quad (30)$$

From (28) it follows that

$$
\begin{bmatrix}
T_2 & 0 \\
0 & I_m
\end{bmatrix}
^{-1}
\begin{bmatrix}
T_1 & 0 \\
0 & I_m
\end{bmatrix}
\begin{bmatrix}
I_r \\
F_1 + L_1 F
\end{bmatrix} = \begin{bmatrix}
R_{21} \\
0
\end{bmatrix} + \hat{U}^{-1} (F_1 + L_1 F) \quad \text{(31)}
$$

Notice that $\hat{U}^{-1} \hat{F} = \begin{bmatrix}
-R_{21}^T \\
0
\end{bmatrix}$ and hence, using (25),

$$
\begin{bmatrix}
R_{21} \\
0
\end{bmatrix} x + \hat{U}^{-1} (F_1 + L_1 F) x = \begin{bmatrix}
R_{21} \\
0
\end{bmatrix} x + \hat{F} x + F_2 R_{11} x = F_2 R_{11} x
$$

for all $x \in V_1$. Combining this with (31), (30) follows easily.