Realization Theory of Discrete-Time Linear Hybrid System

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Abstract: The paper addresses realization theory of discrete-time linear hybrid systems without guards (abbreviated by DTHLS). We present necessary and sufficient conditions for existence of a realization, and a characterization of minimality. In addition, we present basic results on partial realization theory of DTHLSs along with a realization algorithm. We also sketch the application of the obtained results to model reduction and system identification.

Keywords: Realisation theory, minimization, identification, hybrid, identifiability

1. INTRODUCTION

In this paper we present realization theory of discrete-time linear hybrid systems (DTHLS for short), along with a sketch of possible application of the developed theory to system identification and model reduction. The realization problem is one of the central topics of system theory. For DTHLS, its can be stated as follows.

• When is it possible to construct a (preferably minimal) DTHLS state-space representation which generates the specified input/output behavior?
• How to characterize minimal DTHLSs which generate the specified input/output behavior?

Our motivation for developing realization theory is that it is potentially useful for model reduction, systems identification and systems theory. We will elaborate in greater detail on potential applications in §5.

The system class A DTHLS is a discrete-time piecewise-affine systems without guards, i.e. systems where the switching is initiated externally. The system evolves as follows. As long as no discrete event has occurred, the discrete state remains unchanged and the continuous state evolves according to the current linear system. If a discrete event occurs, then the discrete state changes according to the discrete-state-transition map, and the continuous state changes according to the linear reset map. Notice that the discrete events are assumed to be external inputs, and the system has no guards. DTHLSs can be thought of as the discrete-time counterparts of the continuous-time linear hybrid systems without guards of Petreczky (2006); Petreczky and van Schuppen (2008). More precisely, a DTHLS can be obtained from a linear hybrid systems in continuous-time by sampling the latter with a rate Δ and assuming that the time between two discrete events is an integer multiple of Δ.

Motivation The motivation for studying DTHLSs is the following. First, with respect to continuous-time linear hybrid systems, going to discrete-time might make system identification and control synthesis easier. Second, DTHLSs are a subset piecewise-affine hybrid systems, hence any result on realization theory of the latter class has to be consistent with the corresponding results for DTHLSs. In turn, the relevance of piecewise-affine hybrid systems is widely recognized. In addition, a general discrete-time piecewise-affine hybrid systems can be regarded as a feedback interconnection of a DTHLS with an event generating device. Hence, understanding realization theory of DTHLSs should be useful for systems identification and model reduction of piecewise-affine hybrid systems. The latter statement is elaborated upon in §5.

Related work To the best of our knowledge, the results of this paper are new. The realization problem for hybrid systems was first formulated in Grossman and Larson (1995), but no solution was provided. In Paoletti et al. (2007b); Weiland et al. (2006) the relationship between input-output equations and hybrid state-space representations was studied. In Petreczky (2006); Petreczky and Vidal (2007b, 2008) realization theory for various classes of hybrid systems were developed. In particular, the case of continuous-time (bi)linear hybrid systems was already addressed in Petreczky (2006). There is a vast literature on topics related to realization theory, such as system identification, observability and reachability of hybrid systems, see Paoletti et al. (2007a); Vidal et al. (2003); Collins and van Schuppen (2004); Alur et al. (2003).

Outline of the paper §2 presents the definition and basic notions for DTHLSs. §3 presents the main theorems of the paper. §4 sketches the main ideas behind the proof of the results. §5 sketches the potential application of the results to systems identification and model reduction.

2. DISCRETE-TIME LINEAR HYBRID SYSTEMS

We will start by fixing some notation and terminology. Notation Le N be the set of natural numbers including 0. Let Σ be a finite or infinite set which will be referred to as alphabet. Denote by Σ* the set of finite strings or words of elements of Σ, i.e. an element of Σ* is a finite sequence of the form w = a₁a₂⋯ak, where a₁, a₂, ..., a_k ∈ Σ, and
For the empty sequence (word) is denoted by \(v\). The empty sequence \(v\) is the string formed by repeating \(w\) consecutively \(k\) times. The word \(w^k\) is the empty word \(\epsilon\).

### 2.1 Definition of Moore-automaton

Below we will give a brief introduction to the concept of Moore-automaton, see Eilenberg (1974) for more details. Recall that a finite Moore-automaton is a tuple \(A = (Q, \Gamma, O, \delta, \lambda)\) where (1) \(Q\) is the finite state-space, (2) \(\Gamma\) is the finite input alphabet, (3) \(O\) is the output alphabet, (4) \(\delta : Q \times \Gamma \rightarrow Q\) is the state-transition map, (5) \(\lambda : Q \rightarrow O\) is the readout map.

A bijective map \(\phi : Q \rightarrow Q'\) is an automaton isomorphism from \(A\) to \(A'\), denoted by \(\phi : A \rightarrow A'\), if \(\phi\) commutes with the state-transition and readout maps, i.e., \(\phi(\delta(q, \gamma)) = \delta'(\phi(q), \gamma), \text{ and } \lambda(q) = \lambda'(\phi(q)), \) for all \(q, \gamma \in \Gamma\).

### 2.2 Discrete-Time Linear Hybrid System

A discrete-time linear hybrid system, abbreviated by DTHLS, is a discrete-time system of the form

\[
H = \begin{cases} 
  h(t+1) = F(h(t), u(t), d(t)) \\
  z(t) = N(h(t))
\end{cases}
\]

where \(h(t) = (q(t), x(t)) \in \mathcal{H}_H = \cup_{q\in Q} \{q\} \times \mathbb{R}^n\) is the hybrid state at time \(t \in \mathbb{N}\), and \(z(t) = (o(t), y(t)) \in O \times \mathbb{R}^p\) is the output at time \(t\). Here \(Q\) is the finite set of discrete states (modes), \(\mathbb{R}^n = \mathcal{X}_q\) is the continuous state-space associated with a discrete state \(q \in Q\), and \(\mathcal{H}_H = \cup_{q\in Q} \{q\} \times \mathbb{R}^n\) denotes the space of hybrid states. Furthermore, \(O\) is the finite set of discrete outputs and \(\mathbb{R}^p\) is the space of continuous outputs. Furthermore, \(u(t) \in \mathbb{R}^m\) is the continuous input and \(d(t) \in \mathbb{R}^m\) is the discrete input. Here \(\tilde{\Gamma} = \Gamma \cup \{e\}\) is the set of discrete inputs, with \(\Gamma\) being the finite set of discrete events, and \(e\) being a symbol not in \(\Gamma\) describing the situation when no discrete event occurs. The space \(\mathbb{R}^m\) is the set of continuous inputs.

The dynamics of \(H\) can be described as follows. If \(d(t) = e \notin \Gamma\), then the discrete state of \(H\) remains unchanged and the continuous state changes according to the discrete-time linear system associated with the current mode. If \(d(t) \in \Gamma\), then a discrete event \(\gamma = d(t)\) occurred, and the state changes according to the discrete state-transition map and one of the reset maps. For each state \(h = (q, x), q \in Q, x \in \mathbb{R}^n,\) and inputs \(d \in \mathbb{R}^m, u \in \mathbb{R}^m\),

\[
F(h, u, d) = \begin{cases}
(q, A_q x + B_q u) & \text{if } d = e \notin \Gamma, \\
(q, M_{\delta, \gamma} q x + B_q u) & \text{if } d = \gamma \in \Gamma, \delta = \delta(q, \gamma),
\end{cases}
\]

\[
N(h) = (\lambda(q), C_q x)\]

where

- \(\delta : Q \times \Gamma \rightarrow Q\) is the discrete state-transition map
- \(A_q \in \mathbb{R}^{n \times n}, B_q \in \mathbb{R}^{n \times m}\) are the matrices of the state equation of the linear system in mode \(q \in Q\).
- The matrix \(M_{\delta(q, \gamma), \gamma} \in \mathbb{R}^{n \times n}, q \in Q, \gamma \in \Gamma\) is the matrix of the linear reset map associated with the transition from \(q\) to \(\delta(q, \gamma)\) under event \(\gamma \in \Gamma\).
- \(\lambda : Q \rightarrow O\) is the discrete readout map
- The matrix \(C_q \in \mathbb{R}^{p \times n}\) determines the output of the linear system residing in the discrete mode \(q \in Q\).
Definition 4. (Dimension). The dimension \( \text{dim} H \) of \( H \) is defined as a pair of natural numbers; \( \text{dim} H = (\text{card}(Q), \sum_{q \in Q} n_q) \in \mathbb{N} \times \mathbb{N} \), where \( \text{card}(Q) \) denotes the number of discrete states, and \( \sum_{q \in Q} n_q \) is the sum of dimensions of the continuous state-spaces.

For each two pairs of natural numbers \((m, n), (p, q) \in \mathbb{N} \times \mathbb{N} \) define the partial order relation as \((m, n) \leq (p, q)\), if \( m \leq p \) and \( n \leq q \). Notice that with respect to the above ordering, not all pairs are comparable.

Definition 5. (Minimality). \( H \) is a minimal realization of the input-output map \( f \), if \( H \) is a realization of \( f \) and \( \text{dim} H \leq \text{dim} H' \).

Remark 1. Notice that Definition 5 requires the dimension of a minimal DTHLS to be comparable with the dimension of any other DTHLS realization of \( f \). Since we use a partial order on dimensions, the existence of a minimal DTHLS satisfying Definition 5 has to be shown. When it exists, a minimal DTHLS is a minimal element of the set of DTHLS realizations of \( f \), taken with the partial order on dimensions defined above.

Next, we define the notion of DTHLS isomorphisms.

Definition 6. (Isomorphism). Let \( H' \) be the DTHLS \((\tilde{A}', \tilde{R}', \tilde{C}', (\tilde{q}_{\text{init}}, \tilde{x}_{\text{init}}))\) with \( \tilde{A}' = (Q', \Gamma, \delta', \lambda') \), and for all \( q \in Q' \), \( M'_{\tilde{q}} = (A'_{\tilde{q}}, B'_{\tilde{q}}, C'_{\tilde{q}}, \{M'_{\tilde{q}}(\gamma, \gamma') \in \Gamma\}) \).

Let \( H \) be a DTHLS of the form (1). \( H \) and \( H' \) are isomorphic, if there exists a bijection \( S_D : Q \to Q' \), and non-singular matrices \( S_q \in \mathbb{R}^{n_q \times n_q} \), \( q \in Q \) such that

- \( S_D \) is an automorphism isomorphism from \( A \) to \( A' \).
- For all \( q \in Q \) and \( \tilde{q} = S_D(q) \), the dimensions of the continuous state-spaces corresponding to the discrete states \( q \) and \( \tilde{q} \) are equal, i.e. \( n_q = n_{\tilde{q}} \), and \( S_q \) is an isomorphism between the linear systems in \( q \) and \( \tilde{q} \), i.e. \( S_q A_q S_q^{-1} = A'_{\tilde{q}} \), \( S_q B_q = B'_{\tilde{q}} \), \( C_q S_q^{-1} = C'_{\tilde{q}} \).
- The maps \( S_q q' \) commute with the reset maps, i.e. \( \forall \tilde{q} \in \Gamma : S_{\tilde{q}}q_{\tilde{q}} M_{\tilde{q}}(\gamma, \gamma') = M'_{\tilde{q}, \tilde{q}} \).
- The initial states of \( H \) and \( H' \) are related as \( S_D(q_{\text{init}}) = \tilde{q}_{\text{init}} \) and \( S_q x_{\text{init}} = x'_{\text{init}} \).

3. MAIN RESULTS ON REALIZATION THEORY

First we state the main result on minimality of a DTHLS. Second, we formulate necessary and sufficient conditions for existence of a DTHLS realization. Finally, we discuss partial realization theory and realization algorithms. Throughout the section \( f : U^* \to O \times \mathbb{R}^p \) is an input-output map for which a DTHLS realization is sought.

Minimality We can state the following characterization of minimality of DTHLSs.

Theorem 1. (Minimal realization). If \( f \) has a DTHLS realization, then \( f \) has a minimal DTHLS realization. The DTHLS \( H \) is a minimal realization of \( f \), if and only if \( H \) is span-reachable and observable. All minimal DTHLS realizations of \( f \) are isomorphic.

Remark 2. (Rank conditions). Similarly to the continuous-time linear hybrid systems, one can formulate algebraic characterization of observability and span-reachability of DTHLS. This characterization yields numerical algorithms for checking observability and span-reachability and for transforming a DTHLS realization of \( f \) to an observable and span-reachable, hence minimal, DTHLS which realizes \( f \). Hence, there exists a minimization algorithm for DTHLSs.

Existence of a realization We start with defining the notion of hybrid convolution representation of an input-output map, existence of which is a necessary condition for realizability. To this end,

Notation 4. (Discrete and continuous-valued components). For each input-output map \( f \), denote by \( f_C \) the \( \mathbb{R}^p \)-valued part, and by \( f_D \) the \( \mathbb{R}^q \)-valued part of the map \( f \). That is, \( f(v) = (f_D(w), f_C(w)) \in O \times \mathbb{R}^p \) for all \( w \in U^* \).

Remark 3. Recall the definition of \( \tilde{\Gamma} = \Gamma \cup \{e\} \). Every word \( s \in \tilde{\Gamma}^* \) can be uniquely written as

\[
\gamma_1 \gamma_2 \ldots \gamma_k \\ \gamma_k e \gamma_{k+1} \cdots e + 1 \tag{2}
\]

for some \( \gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma, \alpha_1, \alpha_2, \ldots, \alpha_{k+1} \in \mathbb{N}, \) and \( k \geq 0 \). Recall that \( e \) is the word obtained by repeating \( e \) consecutively \( \alpha \) times.

Informally, \( f \) has a hybrid convolution representation if,

(a) \( f_D \) depends only on the discrete events,
(b) \( f_C \) is affine in continuous inputs.

More formally the definition goes as follows.

Definition 7. (Hybrid conv. repr.). The input-output map \( f \) is said to have hybrid convolution representation if for each sequence of discrete events \( v = \gamma_1 \gamma_2 \ldots \gamma_k \in \Gamma^* \), \( \gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma, \) \( k \geq 0 \), there exist sequences \( K_{f_D} : \mathbb{N}^{k+1} \to \mathbb{R}^p \) and \( K_{f_C} : \mathbb{N} \to \mathbb{R}^{p \times m} \) where \( j = 1, 2, \ldots, k+1 \), and a discrete output \( \phi_D(v) \in O \), such that the following holds. Consider a sequence of hybrid inputs

\[
w = (u_1, s_1)(u_2, d_2) \ldots (u_r, d_r) \in U^*
\]

where \( d_i \in \tilde{\Gamma} \) and \( u_i \in \mathbb{R}^m, i = 1, \ldots, r \), and such that the word \( s = d_1 d_2 \ldots d_r \in \tilde{\Gamma}^* \) is of the form (2), and the word \( v = u_1 u_2 \ldots u_r \) formed by the continuous inputs is

\[
u = \gamma_1 \gamma_2 \ldots \gamma_k
\]

where \( \gamma_1 = u_1 u_2 \ldots u_{i-1}, \gamma_2 = u_{i+1} u_{i+2} \ldots u_{i+r}, u_{i+1}, u_{i+2} \ldots u_{i+r} \in \mathbb{R}^m \) and \( u_{i+1}, u_{i+2} \ldots u_{i+r} \in \mathbb{R}^m \) for all \( i = 2, 3, \ldots, k+1 \). In addition, let \( u_{i+1} = 0 \), and let \( \alpha_1, \alpha_2, \ldots, \alpha_{k+1} \in \mathbb{N} \) be the same as in (2) for \( s \). Then,

\[
f_D(w) = \phi_D(w)
\]

\[
f_C(w) = K_{f_C}(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}) + \sum_{j=0}^{\alpha_{k+1}} G_{(\alpha_1+1)}(\alpha_k + j) u_{k+1,j} + \sum_{j=0}^{\alpha_{k+1}} G_{(\alpha_2+1)}(\alpha_k - j, \alpha_{k+1}) u_{k, j} + \sum_{j=0}^{\alpha_{k+1}} G_{(\alpha_{k+1}+1)}(\alpha_k - j, \alpha_{k+1}) u_{1, j}
\]
The role of the sequences $K^f_{\alpha}$ and $G^f_{\alpha,j}$ is best understood by analogy with the linear case. Recall from Callier and Desoer (1991) that the map $y : (\mathbb{R}^n)^* \to \mathbb{R}^p$ is realizable by a linear system, if there exist sequences $G : \mathbb{N} \to \mathbb{R}^{p \times m}$ and $K : \mathbb{N} \to \mathbb{R}^p$, such that $y(u_0 u_1 \cdots u_{t-1}) = K(t) + \sum_{j=0}^{t-1} G(t-j) u_j$. The requirement that $f$ has a hybrid convolution representation is analogous to requiring that $y$ is of the above convolution form, with $K^f_{\alpha}$ and $G^f_{\alpha,j}$ playing the roles of $K$ and $G$.

Next, we define the notion of the Hankel-matrix of an input-output map $f$ admitting a hybrid convolution representation. The entries of the Hankel-matrix will be the input-output map. Next, we define the counterpart of the general Markov-parameters of $f$.

**Definition 9.** (Hankel-matrix). We define the Hankel-matrix for DTHLSs is defined as follows.}

**Definition 8.** (Markov-parameters). Assume that $f$ has a hybrid convolution representation. Define the maps $Z_f : \Gamma^* \to \mathbb{R}^p$ and $Z_{f,j} : \Gamma^* \to \mathbb{R}^p$, $j = 1, 2, \ldots, m$, as follows. For any word $s \in \Gamma^*$ of the form (2),

$$Z_f(s) = K^f_{\alpha}(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}),$$

$$Z_{f,j}(s) = \left\{ G_{v,k+2-l}(\alpha_1-1, \alpha_{l+1}, \ldots, \alpha_{k+1}), \quad \text{if} \ s \notin \Gamma^* \right\} \text{if} \ s \in \Gamma^*;

where $v = \gamma_1 \gamma_2 \cdots \gamma_k$, and if $s \notin \Gamma^*$, i.e., $\alpha_1, i, i = 1, 2, \ldots, k+$ are not all zero, then $l \in \{1, 2, \ldots, m\}$ is such that $\alpha_1 = \alpha_2 = \cdots = \alpha_{i-1} = 0$ and $\alpha_i > 0$, and $e_j$ is the $j$th unit vector in $\mathbb{R}^m$. The collection $(Z_f, Z_{f,j})_{j=1, 2, \ldots, m}$ is the Hankel-matrix of an input-output map of $f$.

The maps $\{Z_{f,j}, Z_{f,j}\}_{j=1, 2, \ldots, m}$ can be thought of as generalizations of Markov parameters for linear systems. They can be computed from the response of $f$ to specific inputs.

**Remark 5.** (Computing the Markov-parameters). For any $s \in \Gamma^*$, one can find input sequences $w_0, w_1, \ldots, w_m \in U^*$ of the same length as $s$, such that $Z_f(s) = f_c(w_0)$ and $Z_{f,j}(s) = f_c(w_j) - f_c(w_0)$, $j = 1, 2, \ldots, m$.

As in the linear case, the Markov-parameters of $f$ can be expressed via the matrices of a DTHLS realizing $f$.

**Proposition 1.** Let $H$ be a DTHLS of the form (1). Then $H$ is a realization of $f$ if and only if $f$ has a hybrid convolution representation and for all $s \in \Gamma^*$ of the form (2), and for all $j = 1, 2, \ldots, m$, the following holds\footnote{In (4), if $l = k + 1$, then the right-hand side of the first equality becomes $C_{q_0} A_{q_0}^{l-1} B_{q_0} e_j$. Likewise, if $k = 0$, then the right-hand side of the second equality becomes $C_{q_0} A_{q_0}^{l-1} x_{init}$.}

$$Z_f(s) = C_{q_0} A_{q_0}^{l-1} B_{q_0} e_j$$
$$\cdots A_{q_0}^{l-1} B_{q_0} e_j$$
$$Z_f(s) = C_{q_0} A_{q_0}^{l-1} B_{q_0} e_j$$
$$\cdots A_{q_0}^{l-1} B_{q_0} e_j$$

$$f_D(\gamma_1 \gamma_2 \cdots \gamma_k) = \lambda(q_k)$$

for where $s \notin \Gamma^*$, $l \in \{1, 2, \ldots, k + 1\}$ is such that $\alpha_l > 0$, and $\alpha_1 = \cdots = \alpha_{l-1} = 0$, $e_j$ is the $j$th unit vector of $\mathbb{R}^m$, $q_0 = q_{init}$ and $q_0 = \delta(q_{\gamma_l+1})$ for each $i = 0, 1, \ldots, k - 1$.

The Hankel-matrix for DTHLSs is defined as follows.

**Definition 9.** (Hankel-matrix). We define the Hankel matrix of $f$, denoted by $H_f$, as the following infinite matrix. Consider the index set $I_f = \{f\} \cup \{1, 2, \ldots, m\}$. The columns of the matrix $H_f$ are indexed by pairs $(s, l) \in \Gamma^* \times I_f$. The rows of the matrix $H_f$ are indexed by pairs of the form $(g, i) \in \Gamma^* \times \{1, 2, \ldots, p\}$. The element of $H_f$ with the row index $(g, i)$ and column index $(s, l)$ is defined as

$$H_f(g, i, (s, l)) = \left\{ \begin{array}{ll}
Z_f(s) & \text{if } l = f \\
Z_{f,j}(s) & \text{if } l = j \in \{1, 2, \ldots, m\}
\end{array} \right.$$
such input-output maps is finite, $H_{f,O}$ has to have finitely many column vectors. Finally, the condition that $W_{fo}$ has finitely many column vectors ensures that $f_D$ is realizable by a Moore-automaton.

**Partial-realization theory** The results above allow us to construct a minimal DTHLS realization from the (infinite number) of Markov-parameters of the map $f$. The question arises what can be computed from finitely many Markov-parameters. This leads to the partial realization problem.

*Definition 12.* (N-partial realization). Assume that $f$ has a hybrid convolution representation. Assume that $H$ is a DTHLS of the form (1). Then $H$ is said to be an N-partial realization of $f$, if (4) holds for all $j = 1, 2, \ldots, m$ and for all $s \in \Gamma^*$ of length at most $N$, i.e. $|s| \leq N$.

Notice that $H$ is a realization of $f$ if and only if $H$ is an N-partial realization of $f$ for all $N \in \mathbb{N}$. Note that $H$ being a N-partial realization involves only a finite number of Markov-parameters of $f$. For the results on partial realization, we need the following notation.

*Notation 5.* Define $H_{f,K,L}$ (resp. $H_{f,O,K,L}$) as the finite matrix formed by the intersection of rows of $H_f$ (resp. $H_{f,O}$), indexed by $(s,i) \in \Gamma^* \times \{1,2,\ldots,p\}$ with $s$ of length at most $K$, and columns of $H_f$, (resp. $H_{f,O}$), indexed by $(v,l) \in \Gamma^* \times \{1,2,\ldots,m\}$ for the case of $H_{f,O,K,L}$, such that $v$ is of length at most $L$. Define the finite table $W_{f,K,L}$, as the sub-table of $W_{fo}$ formed by the intersection of those rows of $W_{fo}$ which are indexed by words of length at most $K$, and those columns of $W_{fo}$ which are indexed by words of length at most $L$. Denote by $\text{card}(H_{f,O,N,N})$ and $\text{card}(W_{f,O,N,N})$ the number of distinct column vectors of $H_{f,O,N,N}$ and $W_{f,O,N,N}$ respectively.

*Theorem 3.* Under conditions on $N \in \mathbb{N}$, similar to the ones for partial realization of linear systems, it is possible to compute a $2N + 1$-partial realization of $f$ from $H_{f,N+1,N}$, $W_{f,N+1,N}$ and $H_{f,O,N+1,N}$. Moreover, if

\[
\begin{align*}
\text{rank} \ H_{f,N,N} &= \text{rank} \ H_f, \\
\text{card}(W_{f,O,N,N}) &= \text{card}(W_{f,O}), \\
\text{card}(H_{f,O,N,N}) &= \text{card}(H_{f,O}),
\end{align*}
\]

then the conditions for existence of a $2N + 1$ partial realization hold, and a minimal (complete, not partial) DTHLS realization of $f$ can be computed from $H_{f,N,N}$ and $W_{f,O,N,N}$. In addition, if $\dim H = (n_d, n_o)$ and $N \geq \max\{n_d, n_o + n_a m\}$, for some DTHLS $H$ such that $H$ is a realization of $f$, then (5) holds.

4. **SKETCH OF PROOF OF THE RESULTS**

Similarly to continuous-time (bi)linear hybrid systems Petreczky (2006), the realization problem for DTHLSs can be solved using the theory of *rational hybrid formal power series* Petreczky (2006). The latter is an extension of the theory of rational formal power series Berstel and Reutenauer (1984).

A *hybrid formal power series* is a combination of classical formal power series and input-output maps of Moore-automata. A *hybrid representation* is an infinite-state Moore-automaton whose state-space has both discrete and continuous components. A hybrid formal power series is *rational*, if it is the input-output map of a hybrid representation, if the latter is viewed as a Moore-automaton.

That is, hybrid formal power series are external behaviors, and hybrid representations are potential state-space representations of these behaviors. With this correspondence in mind, one can formulate a realization problem for hybrid power series and hybrid representations. The solution of that realization problem and the corresponding algorithms are described in Petreczky (2006).

The solution of the realization problem for DTHLSs proceeds then as follows. Find a correspondence between DTHLSs and hybrid representations. Find a correspondence between an input-output map of DTHLSs and a hybrid power series. Furthermore, require that the hybrid representation corresponding to a DTHLS is a state-space representation of the hybrid power series corresponding to the input-output map of that DTHLS. For DTHLSs the correspondence required above is easy to find and the transformation from DTHLSs to hybrid representation and back are computationally effective. In addition, the correspondence between DTHLSs and hybrid representations preserves dimension, systems isomorphism, span-reachability, observability and minimality. This allow us to reduce the realization problem for DTHLSs to the realization problem for hybrid formal power series. The results of this paper follow then from the general results for hybrid power series, by applying the correspondence described above. In addition, the corresponding algorithms for hybrid representations can be used for DTHLSs.

5. **RELEVANCE FOR IDENTIFICATION AND FOR MODEL REDUCTION**

In this section we sketch the significance of the presented results for systems identification and model reduction.

**System identification algorithm** As the subspace identification algorithms van Overschee and Moor (1996) demonstrate, realization theory can yield efficient systems identification algorithms. In fact, our results yield the following identification algorithm for DTHLS. Consider an input-output map $f$. Fix a number $N \in \mathbb{N}$. The choice of $N$ can be based on the estimate on the dimension of a potential realization, or our ability to make measurements for inputs of length $2N + 1$.

1: Compute the Markov-parameters $Z_{f,j}(s)$, $Z_f(s)$ for all $s \in \Gamma^*$ of length at most $2N + 1$ and the values of $f_D(v)$ for all $v \in \Gamma^*$ with $v$ being of length at most $2N + 1$. By Remark 5, this can be done using the responses of $f$ to input sequences of length $2N + 1$. Construct the matrix $H_{f,N+1,N}$, and the tables $W_{f,D,N+1,N}$, $H_{f,O,N,N}$.

2: Compute the $2N + 1$ partial realization of $f$.

It follows from Theorem 3, that if $f$ can indeed be realized by a DTHLS, then for large enough $N$ the procedure above returns a minimal DTHLS realization of $f$. In fact, the right value of $N$ can be chosen based on the dimension of a potential DTHLS realization of $f$.

The above algorithm computes a realization from outputs corresponding to several input sequences. For identification one needs another algorithm which computes a realization of some fixed dimension from the available single time series of measurements, at least for a sufficiently large time series. The authors have not yet succeeded in obtaining such an algorithm. For linear systems, obtaining such algorithms from realization theory took several decades.
Spaces of DTHLSs
Realization theory is useful for studying the geometry and topology of the space of DTHLS. The latter is useful for system identification, fault detection and computer vision Vishwanathan et al. (2007). For linear systems, the topology and geometry of Hankel-matrices were investigated and used in systems identification, see Peeters (1994) and the references therein. We believe that similar results can be obtained for DTHLSs, using the relationship between DTHLSs and (hybrid) formal power series, and by further extending the results of Sontag (1987); Petreczky and Vidal (2007a).

Model reduction
First, the minimization algorithms for DTHLSs themselves represent a primitive model reduction method. Second, partial-realization theory can also be used for model reduction, by extending the moment matching method, see Antoulas and Sorensen (2001), to DTHLSs. The core of moment matching is to approximate a high-order system with a lower order one, such that the lower order system is a partial realization of a finite number of Markov-parameters of the original system. For DTHLSs the role of Markov-parameters (or moments) is played by the generalized Markov-parameters.

Identifiability for DTHLSs
The results on existence and uniqueness of minimal DTHLS realizations are useful for studying identifiability of DTHLSs. I.e. for the structural identifiability of a parametrization of minimal DTHLSs, it is necessary and sufficient that no two different parameters yield isomorphic systems. One can then extend van den Hof (1998) to study structural identifiability.

Piecewise-affine hybrid systems
It is easy to see that a discrete-time piecewise-affine system with guards and linear (affine) reset maps (abbreviated by PAHS) can be thought of as the output feedback interconnection of a DTHLS with an event generator. Hence, a necessary condition for minimality of a PAHS is that its DTHLS component is minimal. Furthermore, we can replace the DTHLS component of a PAHS with a minimal one, without changing the input-output behavior of the PAHS. This implies that model reduction algorithms for DTHLSs can be used for model reduction of PAHSs.

In addition, a necessary condition for identifiability of a parametrization of PAHSs is that no two distinct parameters yield PAHSs whose DTHLS components realize the same behavior. If the latter condition is violated, then the corresponding PAHSs will also yield the same input-output behavior, and hence the parametrization will not be identifiable. That is, (structural) identifiability of the DTHLS components is a necessary condition for (structural) identifiability of PAHSs. Hence, the results of the paper are relevant for identifiability of PAHSs.

6. CONCLUSIONS AND FUTURE WORK
We have sketched realization theory of discrete-time linear hybrid systems along with its potential applications to systems identification and model reduction. Topics of further research include realization theory for piecewise-affine systems with guards, and application of the presented results to system identification and model reduction.

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REFERENCES


