Abstract: The paper address the problem of minimality and identifiability of Switched ARX (abbreviated by SARX) models. We propose a notion of identifiability and minimality for SARX models which depends only on the parameters of the model, not on data. We formulate conditions for minimality and identifiability of SARX systems. In particular, we show that SARX systems are generically identifiable.

Keywords: switched ARX, minimization, identification, hybrid, identifiability

1. INTRODUCTION

The paper deals with the problem of identifiability and minimality of switched ARX systems.

Motivation. Switched ARX systems (abbreviated as SARX) are popular in hybrid systems community, due to their simplicity and modelling power. In particular, most of hybrid system identification algorithms were developed for switched ARX systems. Despite their popularity, identifiability and minimality of switched ARX are not yet completely understood.

Minimality and identifiability are essential for analyzing algorithms for identification and adaptive control. Indeed, only identifiable parameterizations have the chance to be identified correctly by a parameter estimation algorithm. For this reason, identifiability is usually one of the conditions for correctness of parameter estimation algorithms. Minimality is closely related to identifiability: parameters which occur only in the non-minimal system component are not identifiable.

Contribution of the paper. We propose a definition and conditions for minimality and identifiability of SARX systems. Minimality and identifiability depend only on the system parameters and not on the data generated by the system. The conditions for minimality and identifiability can be checked by numerical algorithms. We also show that minimality is closely related to identifiability. In addition, we show that SARX parameterizations are generically minimal and are generically identifiable.

In order to prove these results, we convert SARX systems to a state-space form, using the regressors as a state-variable. We then analyze minimality and identifiability of the resulting state-space representation by applying Petreczky et al. (2010). However, the results of the paper are not trivial consequences of Petreczky et al. (2010). This is due to the rich structure of SARX systems which allows us to derive stronger results than for general linear switched systems. For example, the relationship between minimality and identifiability is much more direct for SARX systems than for linear switched systems.

It is worth noting that a SARX system can be minimal or identifiable, even if none of the ARX subsystems is minimal (resp. identifiable). In fact, the relationship between identifiability and minimality of SARX systems and their ARX subsystems is not straightforward.

Related work. Identification of hybrid systems is an active research topic, see for example, Bako et al. (2009b,a); Vidal (2008); Verdult and Verhaegen (2004); Juloski et al. (2005b); Roll et al. (2004); Ferrari-Trecate et al. (2003); Nakada et al. (2005); Weiland et al. (2006); Paoletti et al. (2007b); Flies et al. (2008) and the overview Juloski et al. (2005a); Paoletti et al. (2007a) on the topic. Many of the major contributions are formulated only for SARX systems, Vidal (2008); Ferrari-Trecate et al. (2003); Lauer et al. (2009); Paoletti et al. (2010). The relationship between SARX systems and state-space representations was addressed in Paoletti et al. (2010); Weiland et al. (2006), and in this paper we use some of those results.

To the best of our knowledge, the results of the paper are new. However, Vidal (2008) contains persistence of excitation conditions for SARX systems, which is related to identifiability. The main difference between the two concepts is that the former is a property of the data, while the latter is a property of the parameterization. Vidal (2008) also proposes a definition of minimality of SARX which implies our definition. However, the two definitions are not equivalent.

The technical report represents an extended version of the current paper.

Outline. In §2 we define SARX systems and the corresponding systems theoretic concepts such as minimality and identifiability. In §3 we present the transformation of SARX systems into state-space form. In §4 we discuss the relationship between minimality of a SARX system and its subsystems. In §5 we provide sufficient conditions
for strong minimality. In §6 we discuss the relationship between minimality and identifiability. In §7 we show that minimality and identifiability are generic properties.

2. DEFINITION OF SARX SYSTEMS

Throughout the paper, p will denote the output dimension and m will denote the input dimension. The set Q will denote the set of discrete modes, and without loss of generality, we assume that Q = {1, ..., D}. Denote by T = N the time-axis of natural numbers.

The definition of SARX systems is as follows.

Definition 1. (SARX systems). A SARX system S of type (n_y, n_u), where 0 < n_u ≤ n_y are integers, is a collection S = \{n_q\}_{q \in Q}, where n_q, q \in Q are p × (n_y p + n_u m) matrices.

We will call a SARX system a SISO SARX system if p = m = 1.

Notation 1. In the sequel we will use the following decomposition for the matrices n_q:

n_q = \begin{bmatrix} n_q^1 & n_q^2 & \cdots & n_q^{n_u n_q} \\ \end{bmatrix},

where n_q^j \in \mathbb{R}^{p \times p}, i = 1, \ldots, n_y, n_q^j \in \mathbb{R}^{p \times m}, j = n_y + 1, \ldots, n_u + n_y.

In order to assign semantics to SARX systems defined above, we need to formalize the concept of input-output behavior for switched systems, of which SARX systems form a subclass. To this end, we will need the notions of hybrid input and input-output map.

Notation 2. Denote by U = Q × \mathbb{R}^m. We denote by U^+ the set of all non-empty and finite) sequences of elements of U. A sequence

w = (q_0, u_0) \cdots (q_t, u_t) \in U^+, t \geq 0

(1)

describes the scenario, when the discrete mode q_i and the continuous input u_i are fed to the system at time t, for i = 0, ..., t.

The input-output behaviors of interest are then maps f : U^+ → \mathbb{R}^p. The value f(w) describes the output of the system at time t, generated as a response of the system to the hybrid input w. Now we are ready to define the semantics of SARX systems. Unlike systems in state-space form, SARX systems describe the relationship between the inputs and outputs of switched system directly.

Definition 2. (Input-output maps of SARX systems). The SARX S is a realization of the input-output map f, if for all w ∈ U^+ of the form (1), the outputs

y_i = f((q_0, u_0) \cdots (q_i, u_i)), i = 0, \ldots, t

satisfy the equation

y_i = n_q \phi_i

(2)

where we define the regressor \phi_i \in \mathbb{R}^{(n_y p + n_u m)} as

\phi_i = \begin{bmatrix} y_{t-1} & y_{t-2} & \cdots & y_{t-n_y} & u_{t-1} & \cdots & u_{t-n_u} \end{bmatrix}^T,

and for all j < 0, we set y_j = 0 and u_j = 0.

Two SARXs are called equivalent, if they are realizations of the same input-output map.

Definition 3. (Dimension). The dimension of a SARX system S of type (n_y, n_u) is the number p n_y + n_u m and it is denoted by dim S.

Definition 4. (Minimality). A SARX S of is minimal, if there exists no equivalent SARX of dimension less than dim S.

In order to be able to speak of identifiability, we need the notion of parameterization of SARX systems.

Notation 3. Denote by SARX(n_y, n_u, m, p, Q) the set of all SARX systems of type (n_y, n_u) with input space \mathbb{R}^m, output space \mathbb{R}^p, and set of discrete modes Q.

Definition 5. (Parametrization) Assume that Θ ⊆ \mathbb{R}^d is the set of parameters. A SARX parameterization is a map

\Pi : Θ → SARX(n_y, n_u, m, p, Q)

(4)

Definition 6. (Identifiability) The parameterization \Pi is called identifiable, if for \theta_1 \neq \theta_2 \in Θ, the corresponding SARX \Pi(\theta_1) and \Pi(\theta_2) are not equivalent.

The intuition behind the above definition is that if a parameterization is not identifiable, then there might exists different parameter values which yield the same observed behavior and hence they cannot be distinguished from each other by input-output experiments.

3. CONVERTING SARX SYSTEMS INTO STATE-SPACE FORM

In order to present the transformation of a SARX to a state-space system, we have to recall the definition of discrete-time linear switched systems. For a more detailed exposition, see Sun and Ge (2005b); Liberzon (2003); Petreczky et al. (2010).

Definition 7. A linear switched system (abbreviated by DTLS) is a discrete-time system \Sigma represented by

\begin{align*}
x_{t+1} &= A_q x_t + B_q u_t \\
y_t &= C_q x_t,
\end{align*}

(5)

Here x_t ∈ \mathbb{R}^n is the continuous state at time t ∈ T, u_t ∈ \mathbb{R}^m is the continuous input at time t ∈ T, y_t ∈ \mathbb{R}^p is the continuous output at time t ∈ T, q_t ∈ Q is the discrete mode (state) at time t, Q is the finite set of discrete modes of \Sigma. For each discrete mode q ∈ Q, the corresponding matrices are of the form A_q ∈ \mathbb{R}^{n \times n}, B_q ∈ \mathbb{R}^{n \times m} and C_q ∈ \mathbb{R}^{p \times n}. The number n is called the dimension of Σ, and it is denoted by dim Σ.

Notation 4. We will use

(p, m, n, Q, \{A_q, B_q, C_q\} | q ∈ Q)

as a short-hand notation for DTLSs of the form (5).

For any hybrid input sequence w ∈ U^+ of the form (1), denote by y_G(w) the corresponding output y. We call the map

y_G : U^+ → \mathbb{R}^p

the input-output map of Σ. An input-output map of Σ is said to be realized by the DTLS Σ if the input-output map of Σ coincides with f. The DTLS Σ is a minimal realization, if for any DTLS realization \hat{Σ} of y_G, dim \hat{Σ} ≤ dim Σ.

In order to present the state-space representation of SARX systems, we use the following notation.
\textbf{Definition 8.} Let $S = \{n_q\}_{q \in Q}$ be a SARX system of type $(n_y, n_u)$. The DTLSS $\Sigma_S$ associated with a SARX $S$ is defined as $\Sigma_S = (p, m, n, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$, where $n = pn_y + mn_u$, and

$$A_q = \begin{bmatrix} A_y^q, A_u^q \end{bmatrix},$$

$$A_q^u = \begin{bmatrix} \begin{bmatrix} n_q^1, n_q^{n_y-1} & n_q^n \end{bmatrix} \end{bmatrix},$$

$$A_q^y = \begin{bmatrix} I_{(n_y-1)p} & O_{(n_y-1)p \times p} \\ O_{m \times p(n_y-1)} & O_{m \times p} \end{bmatrix},$$

$$A_q^u = \begin{bmatrix} O_{n_q n_y + 1} & \cdots & O_{n_q n_y + n_y - 1} \\ O_{n_q n_y + n_y + 1} & \cdots & O_{n_q n_y + 2} \end{bmatrix},$$

$$C_q = n_q,$n

$$B_q = \begin{bmatrix} O_{p n_y \times m} \\ O_{m \times (n_u - 1) \times m} \end{bmatrix}$$

(6)

where we used the decomposition of $n_q$ from Notation 1.

\textbf{Lemma 1.} (Weiland et al. (2006)). The SARX $S$ is a realization of the input-output map $f$ if and only if the associated DTLSS $\Sigma_S$ is a realization of $f$.

The main idea behind the proof of Lemma 1 is that for all $t \in T$, the state $x_t$ of $\Sigma_S$ equals the regressor $\phi_t$ from (3).

The lemma above allows us to reduce the problem of identifiability and minimality for SARX systems to that of DTLSSs of the form (6). This will be done in the subsequent sections.

Finally, we note that none of the linear subsystems of $\Sigma_S$ is minimal. However, as we shall see later, the system $\Sigma_S$ is generically minimal.

\textbf{Lemma 2.} (Non-minimality). Consider the DTLSS $\Sigma_S$. For every $q \in Q$, the linear system $(C_q, A_q, B_q)$ is not observable and hence $(C_q, A_q, B_q)$ is not minimal.

The proof of Lemma 2 can be found in Appendix A.

4. MINIMALITY OF SARX SYSTEMS

In this section we will analyze minimality of SARX systems. We start by stating a number of simple properties of minimal SARX systems. After that, we link minimality of SARX systems with the minimality of the associated DTLSSs systems. This allows us to formulate sufficient conditions for minimality of SARX systems.

4.1 Minimality: elementary conditions

Below we will state some elementary properties of minimal SARX systems.

\textbf{Lemma 3.} If the SISO SARX system $S$ is minimal, then there must exist $q, \bar{q} \in Q$ such that $n_q^{n_y + n_u} \neq 0$.

The proof of Lemma 3 can be found in Appendix A. Recall that in the classical literature, a SISO ARX is said to be minimal if and only if the numerator and the denominator of its transfer function are co-prime polynomials. Below we show that our definition of minimality is consistent with the traditional one.

\textbf{Lemma 4.} A SISO ARX system is minimal according to Definition 4 if and only if the numerators and denominators of its transfer function are co-prime.

The proof of Lemma 4 can be found in Appendix A. One might have noticed that our definition of minimality is ambiguous, since an input-output map could have two minimal SARX realizations of type $(n_y, n_u)$ and $(\bar{n}_y, \bar{n}_u)$ respectively with $(n_y, n_u) \neq (\bar{n}_y, \bar{n}_u)$. According to the lemma below that this is impossible in the SISO case.

\textbf{Lemma 5.} If at least one of the ARX subsystems of a SISO SARX system is minimal, the system is minimal.

The proof of Lemma 5 can be found in Appendix A. One might have noticed that our definition of minimality is ambiguous, since an input-output map could have two minimal SARX realizations of type $(n_y, n_u)$ and $(\bar{n}_y, \bar{n}_u)$ respectively with $(n_y, n_u) \neq (\bar{n}_y, \bar{n}_u)$. According to the lemma below that this is impossible in the SISO case.

\textbf{Lemma 6.} Assume that $S_1$ and $S_2$ are two minimal and equivalent SISO SARX systems such that $S_1$ is of type $(n_y, n_u)$ and $S_2$ is of type $(\bar{n}_y, \bar{n}_u)$. Then $(n_y, n_u) = (\bar{n}_y, \bar{n}_u)$.

The proof of Lemma 6 can be found in Appendix A.

4.2 Minimality of SARX and of the associated state-space representations

In this section we analyze the minimality of the DTLSSs associated with SARX systems. The motivation for this lies in the following corollary of Lemma 1.

\textbf{Corollary 1.} If the associated DTLSS $\Sigma_S$ is a minimal realization, then $S$ is a minimal realization.

The proof of Corollary 1 can be found in Appendix A. Corollary 1 allows us to reduce the problem of minimality of SARX to that of DTLSSs, the latter has already been investigated in Petreczky et al. (2010). It prompts us to propose the following definition.

\textbf{Definition 9.} A SARX system $S$ is called strongly minimal, if the corresponding DTLSS $\Sigma_S$ is minimal.

By Corollary 1, strong minimality implies minimality.

We conclude by presenting a number of examples which clarify the relationship between (strong) minimality of SARX systems and minimality of ARX subsystems.

\textbf{Example 1.} Consider the SARX system $S$ with discrete modes $Q = \{1, 2\}$ such that the ARX system associated with mode 1 is

$$y_t = -y_{t-2} + u_{t-1}$$

and the ARX system associated with mode 2 is

$$y_t = -2y_{t-2} + 2u_{t-1}.$$
Minimality of the ARX subsystems is not necessary for strong minimality (and hence minimality) of the whole system.

Example 2. Consider again the SARX system $S$ with two discrete modes $Q = \{1, 2\}$ such that the ARX system in mode 1 is of the form

$$y_t = 8y_{t-1} - 15y_{t-2} + u_{t-1} - 3u_{t-2},$$

and the ARX system in mode 2 is of the form

$$y_t = y_{t-1} + 2y_{t-2} + u_{t-1} + u_{t-2}.$$  

The transfer function of the ARX in the first mode is

$$\frac{8z - 15}{z^2 - 2z + 1},$$

and the transfer function of the second ARX is

$$\frac{1}{z^2},$$

hence neither of them is minimal. Yet, the DTLSS $\Sigma_S$ can easily seen to be minimal, using the conditions of Petreczky et al. (2010). Since strong minimality implies minimality of SARX systems, we get that $S$ is minimal.

5. SUFFICIENT CONDITIONS FOR MINIMALITY

As we have seen in the previous section, strong minimality implies minimality. By Petreczky et al. (2010) strong minimality and hence minimality can be checked algorithmically. Indeed, strong minimality of a SARX system $S$ means minimality of the associated DTLSS $\Sigma_S$. The latter can be checked by checking if the rank of each of the finite matrices $R(\Sigma_S)$ and $O(\Sigma_S)$ defined in (Petreczky et al., 2010, Theorem 2) equals the dimension of $\Sigma_S$.

We can also formulate sufficient conditions for minimality which do not involve computing DTLSSs.

Theorem 1. (Sufficient conditions for strong minimality). Consider a SISO SARX system $S = (n_y, n_u)$ of type $(n_y, n_u)$. For all modes $q, \tilde{q} \in Q$ define the polynomials

$$\chi_q(z) = z^{n_y} - \sum_{j=1}^{n_y} n_y^j z^{n_y-j},$$

$$v_q(z) = \sum_{i=1}^{n_u} n_u^i z^{n_u-i},$$

$$\phi_{\tilde{q}, q}(z) = \sum_{j=1}^{n_u} n_u^j z^{n_u-j},$$

where $\psi_{\tilde{q}, q,j}(z)$ is defined recursively for $j = 0, 1, 2, \ldots$, as follows: $\psi_{\tilde{q}, q,0}(z) = 1$ and

$$\psi_{\tilde{q}, q,j+1}(z) = z \psi_{\tilde{q}, q,j}(z) + (n_q - n_q) \phi_j,$$

where the vectors $d_j \in \mathbb{R}^n_u$ are defined as follows: $d_0 = e_1$ and if $d_j = (d_{j,1}, \ldots, d_{j,n_y}, 0, \ldots, 0)^T$, $d_{j+1,1}, \ldots, d_{j,n_y} \in \mathbb{R}$, then $d_{j+1} = (n_u d_{j,1} + d_{j,n_y} - 1, 0, \ldots, 0)^T$.

Then $S$ is strongly minimal, if the following conditions hold

(A) there exists discrete modes $q_0$ and $q_1$ such that the polynomials $\chi_{q_1}(z)$ and $\phi_{q_0,q_1}(z)$ are co-prime, and

(B) there exists discrete modes $q_2$ and $q_3$, such that $v_{q_2}(z)$ and $\chi_{q_3}(z)$ are co-prime, $n_{q_2} + n_{q_3} \neq 0$ and $n_{q_2} \neq n_{q_3}$.

The proof of Theorem 1 can be found in Appendix A. Theorem 1 is analogous to the well-known result that if a SISO transfer has no zero-pole cancellation (i.e. its numerator and denominator are coprime) and its denominator is of degree $n$, then all its minimal realizations are of order $n$.

6. IDENTIFIABILITY OF SARX SYSTEMS

In this section we study identifiability of SARX systems. We derive our results by reducing identifiability analysis of SARX systems to that of the associated DTLSSs. This is possible due to the following corollary of Lemma 1.

Corollary 2. A SARX parametrization $\Pi$ is identifiable, if and only if the DTLSS parametrization $\Pi_{sw}: \Theta \ni \theta \mapsto \Sigma_{\Pi(\theta)}$ is identifiable.

The proof of Corollary 2 can be found in Appendix A.

In order to apply the results of Petreczky et al. (2010), we have to restrict attention to strongly minimal SARX systems. To this end, we need the following terminology.

Definition 10. (Minimality of parametrizations) The parametrization $\Pi$ is called minimal (resp. strongly minimal), if for all $\theta \in \Theta$, $\Pi(\theta)$ is minimal (resp. strongly minimal).

If a SARX parametrization is strongly minimal, then the corresponding DTLSSs parametrization will be minimal. Hence, we can apply the conditions and algorithms described in Petreczky et al. (2010) for analyzing the identifiability if the latter parametrization. By Corollary 2 the identifiability of the latter parametrization is identifiability of the original SARX parametrization.

In fact, for the SISO case (i.e. when $p = m = 1$), we can derive even stronger results, by showing minimality is sufficient for identifiability. To this end, we need the following definition.

Definition 11. (Injective parametrizations) An SARX parametrization $\Pi$ is said to be injective if $\Pi$ is an injective map.

An injective parametrization allows us to exclude the situation where two different parameter values lead to the same SARX system. The ARX parametrization $y_t = \theta y_{t-1} + u_{t-1}$ with $\theta \in \mathbb{R}$ is not injective, since any $\theta$ and $-\theta$ always lead to the same ARX system.

1 See Petreczky et al. (2010) for the definition of a parametrization and identifiability of DTLSSs.

2 See Petreczky et al. (2010) for the definition.
The following theorem, which is one of the main results of the paper, describes the relationship between strong minimality and identifiability.

**Theorem 2.** Assume that \( p = m = 1 \). If a SISO SARX parametrization \( \Pi \) is injective and strongly minimal, then \( \Pi \) is identifiable.

The proof of Theorem 2 can be found in Appendix A. In order to prove Theorem 2, we will need the following result which is interesting on its own right.

**Theorem 3.** Consider two SISO SARX systems \( S_1 = \{n_q \}_{q \in G} \) and \( S_2 = \{h_q \}_{q \in G} \) of type \( (n_q, \nu) \) and \( (h_q, \nu) \) and assume that for some \( q \in G \), either \( n_q \neq 0 \) or \( n_q + \nu \neq 0 \). If there exists an isomorphism \(^3\) between the associated DTLSSs \( \Sigma_{S_1} \) and \( \Sigma_{S_2} \), then this isomorphism is the identity map.

The proof of Theorem 3 can be found in Appendix A. Theorem 3 implies that under some mild conditions that the transformation of two different SARX systems to state-space representations cannot result in isomorphic systems. Recall that strong minimality of a SARX system does not imply minimality of its ARX subsystems. Similarly, identifiability of a SARX parametrization does not imply the identifiability of the corresponding parametrization of ARX subsystems. The example below demonstrates this point.

**Example 4.** Consider the SARX parametrization \( \Pi \) with \( \Theta = \mathbb{R}^2 \), and consider the parametrization \( \Pi((\theta_1, \theta_2)) = \{n_q(\theta_1, \theta_2) \}_{q \in G} \), where

\[
\begin{align*}
n_1 &= [(\theta_1 + \theta_2) - \theta_1 \theta_2, 1 - \theta_2] \\
n_2 &= [(2 + \theta_2) - 2\theta_2, 1 - \theta_2]
\end{align*}
\]

Define the set \( G = \{(\theta_1, \theta_2) \mid \theta_1 \neq 2 \} \). Consider the restriction \( \Pi_G \) of \( \Pi \) to \( G \). Using Theorem 1 one can check that for any \( (\theta_1, \theta_2) \in G \), the SARX system \( \Pi((\theta_1, \theta_2)) \) is strongly minimal. Hence, the parametrization \( \Pi_G \) is identifiable by Theorem 2. Identifiability of \( \Pi_G \) can also be checked by considering the switching sequence 112 and input \( u_0 = 1, u_t = 0, t > 0 \) and noticing that then \( y_0, y_1 = 1, y_2 = \theta_1, y_3 = 2\theta_1 + \theta_2 \theta_1 - 2\theta_2 \) from which \( \theta_2 = \frac{(y_3 - 2y_2)}{\theta_1} \). Hence, \( \theta_1 \) and \( \theta_2 \) can be determined from the outputs \( y_2 \) and \( y_3 \).

Note however, that for any \( (\theta_1, \theta_2) \), the ARX subsystems of \( \Pi((\theta_1, \theta_2)) \) are not identifiable, since their dynamics does not depend on \( \theta_2 \).

### 7. Minimality and Identifiability Are Generic

In the previous sections we have established that strong minimality is sufficient for minimality and that it is also sufficient for identifiability. However, we have also demonstrated that for some minimal SARX systems strong minimality does not hold. Hence, one may wonder how typical strong minimality is.

Below we will show that strong minimality is a generic property, i.e. it holds for almost all SARX systems, if \( |Q| > 1 \). This also means that identifiability is a generic property. In other words, strong minimality occurs very frequently.

In order to formalize these results, we need the following terminology.

**Definition 12.** (Genericity). A subset \( G \) of \( \Theta \subset \mathbb{R}^d \) is generic, if \( G \) is non-empty and there exists a non-zero polynomial \( P(x_1, \ldots, x_d) \) in \( d \) variables such that \( G = \{ \theta \in \Theta \mid P(\theta) \neq 0 \} \).

That is, a generic subset of \( \Theta \) is a non-empty subset whose complement in \( \Theta \) satisfies a polynomial equation.

**Definition 13.** (Generic Identifiability and Minimality) The parametrization \( \Pi \) is said to be generically identifiable if there exists a generic subset \( G \) of \( \Theta \), such that the parametrization \( \Pi_G : G \ni \theta \mapsto \Pi(\theta) \) is identifiable.

Similarly, \( \Pi \) is generically minimal (respectively generically strongly minimal), if there exists a generic subset \( G \) of \( \Theta \), such that the parametrization \( \Pi_G : G \ni \theta \mapsto \Pi(\theta) \) is minimal (respectively strongly minimal).

**Definition 14.** (Polynomial parametrization) Let \( K = \{my + mnu\}_{Q} \). Then any SARX system of type \( (n_q, \nu) \) can be identified with a point in \( \mathbb{R}^K \), by identifying the system with its parameters \( \{n_q\}_{q \in G} \). Thus, SARX \( (n_q, \nu, m, p, Q) \) can be identified with the space \( \mathbb{R}^K \).

A parametrization \( \Pi \) is said to be polynomial, if \( \Theta \) is an affine algebraic variety and \( \Pi \) is a polynomial map from \( \Theta \) to \( \text{SARX}(n_q, \nu, m, p, Q) \).

Intuitively, if a property is generic for a parametrization, then every member of the parametrization can be approximated with arbitrary accuracy by another member which has this property. Another interpretation is that if we randomly generate parameters, then the property will hold for the thus obtained random parametrization with probability one.

**Example 5.** Consider the parametrization \( \Pi \) from Example 4. The set \( G \) from Example 4 is generic. Hence, since the parametrization \( \Pi_G \) is strongly minimal and identifiable, the parametrization \( \Pi \) is generically strongly minimal, generically minimal, and generically identifiable.

**Theorem 4.** (Generic minimality). If \( |Q| > 1 \), \( \Pi \) is a polynomial parametrization and \( \Pi \) contains a strongly minimal SARX system, (i.e. for some \( \theta \in \Theta \), \( \Pi(\theta) \) is strongly minimal), then \( \Pi \) is generically minimal.

The proof of Theorem 4 can be found in Appendix A. Notice that Theorem 2 implies the following corollary.

**Corollary 3.** Consider the SISO case, i.e. \( p = m = 1 \). If a SARX parametrization is injective, it is polynomial, and it is generically strongly minimal, then it is generically identifiable.

The proof of Corollary 3 can be found in Appendix A. Corollary 3 and Theorem 4 yield the following.

**Corollary 4.** Assume that \( p = m = 1 \). If a SISO SARX parametrization is polynomial and it contains a strongly minimal element, then it is generically identifiable.

The proof of Corollary 4 can be found in Appendix A.

The trivial SISO SARX parametrization \( \Pi_{triv} \) is the SARX parametrization defined as follows: \( \Theta = \mathbb{R}^{Q(n_q + n_r)} \)

---

\(^3\) See Petreczky et al. (2010) for the definition of isomorphism between DTLSSs.
and $\Pi_{rew}$ is the identity map. From Corollary 3-4 we obtain that

Corollary 5. The trivial parametrization is generically minimal and in the SISO case it is generically identifiable.

The proof of Corollary 5 can be found in Appendix A.

8. CONCLUSIONS

In this paper we proposed definitions for minimality and identifiability of SARX systems. We have shown that minimal SARX parametrizations are also identifiable. We have also formulated sufficient and necessary conditions for minimality of SARX systems.

Future research is aimed at finding a more complete characterization of minimality and extending the results to MIMO systems and systems with autonomous switching.

REFERENCES


Appendix A. PROOFS

Proof. [Proof of Lemma 2] Notice that for each $q \in Q$, $A_q$ contains a zero row, hence rank$A_q < n_u + n_y$. This means that $\lambda = 0$ is an eigenvalue of $A_q$. By the Hautus-criterion, $(C_q, A_q)$ is an observable pair if and only if the matrix $[C_q T, A_q T]$ has rank $n_u + n_y$ for all the eigenvalues of $A_q$. We will show that for $\lambda = 0$ this matrix cannot be of full row rank. Indeed, for $\lambda = 0$ the matrix becomes $[C_q^T, -A_q T]$. But $C_q$ equals the first row of $A_q$ multiplied by $-1$. Hence, $[C_q^T, -A_q T]$ will have the same rank as $A_q$ and that is smaller than $n_u + n_y$.

Proof. [Proof of Lemma 3] Assume the contrary, i.e. that $n_q^{n_u+n_y} = 0$ for all $q \in Q$. Define the vectors $\hat{n}_q \in \mathbb{R}^{n_u+n_y-1}$, $q \in Q$.

$$\hat{n}_q = [n_1 \ldots n_{n_u+n_y-1}]$$

Define the regressors $\hat{\phi}_t$ as

$$\hat{\phi}_t = \begin{bmatrix} y_{T-1} \ldots y_{T-n_y} u_{T-1} \ldots u_{T-n_y-n_u} \end{bmatrix}^T,$$

where we used the convention that $y_0 = 0$ and $u_j = 0$ for $j < 0$. It then follows that

$$y_t = n_q \hat{\phi}_t = \hat{n}_q \hat{\phi}_t$$

for all $t \in T$. Hence, $\hat{S} = \{\hat{n}_q \mid q \in Q\}$ realizes the same input-output map as $S$. But the dimension of $\hat{S}$ is smaller than that of $S$, which contradicts the minimality of $S$.

Proof. [Proof of Lemma 4] The proof follows from the classical linear theory, by observing that two ARX systems realize the same input-output map if and only if they have the same transfer function (modulo zero/pole cancelation).

Consider an ARX system $y_t = n_q \phi_t$ and assume that it is minimal. If its transfer function admits a zero-pole cancellation, then the degrees of the numerator and denominator of the transfer function decrease by one. The latter means that the transfer function can be realized by an ARX of type $(n_y-1, n_u-1)$. The dimension of the latter is $n_y + n_u - 2$ and hence smaller than that of the original system, which was supposed to be minimal. Moreover, this new ARX system will realize the same input-output map as the original one.
Conversely, consider an ARX system $S$ whose transfer function does not allow zero/pole cancellation. Let $f$ be the input-output map of $S$ and assume that the ARX system $S$ is a minimal realization of $f$. Then the transfer function $H_S(z)$ cannot allow a zero/pole cancellation and it must be equal to the transfer function $H_S(z)$ of $S$. Since neither $H_S(z)$ nor $H_S(z)$ allow zero/pole cancellation, their equality implies the equality of the numerators and denominators respectively, viewed as polynomials. In particular, the corresponding coefficients are the same and hence the parameters of the two ARX systems are the same too. In particular, the dimensions of the two systems will be the same, and hence $S$ is then a minimal realization of its input-output map.

**Proof.** [Proof of Lemma 5] Consider $S = \{(n_q)_{q \in Q}\}$ and assume that for some $q_n \in Q$, the ARX $y_t = n_{q_n} \phi_t$ is minimal. Assume that $S$ is not minimal and hence there exists a SARX $S_m = \{(n_q)_{q \in Q}\}$ such that $\dim S_m \leq \dim S$ and $S_m$ realizes the same input-output map as $S$. It then follows that the dimension of the ARX $y_t = n_{q_n} \phi_t$ is larger than that of $y_t = n_{q_n} \phi_t$. It also follows that both $y_t = n_{q_n} \phi_t$ and $y_t = n_{q_n} \phi_t$ realize the same linear input-output map. This contradicts the minimality of $y_t = n_{q_n} \phi_t$.

**Proof.** [Proof of Lemma 6] Pick any discrete state $q$ and consider the transfer functions $H_1(z), i = 1, 2$ of the ARX system in mode $q$ associated with the SARX $S_{1,i}, i = 1, 2$. Since $S_1$ and $S_2$ are equivalent, they produce the same response to any input if the discrete mode is kept to be $q$. Hence, the ARX systems corresponding to the mode $q$ are also equivalent, i.e., $H_1(z)$ and $H_2(z)$ describe the same input-output behavior. This means that the transfer functions $H_1(z)$ and $H_2(z)$ are equal as rational expressions, after possibly performing zero/pole cancellation. The degrees of the numerators of $H_1(z)$ and $H_2(z)$ are respectively $n_1$ and $n_1$ and the degrees of the denominators of $H_1(z)$ and $H_2(z)$ are respectively $n_1$ and $n_1$. Performing zero/pole cancellation does not change the difference between the degree of the numerator and the degree of the denominator. Hence, we obtain that $n_1 - n_1 = n_1 - n_1$ must hold. But since both $S_1$ and $S_2$ are minimal SARX realizations of the same input-output map, their dimensions must agree and hence $n_1 + n_1 = n_1 + n_1$. It is easy to see that the only solution to the system of equations

\[
\begin{align*}
&n_1 - n_1 = n_1 - n_1 \\
&n_1 + n_1 = n_1 + n_1
\end{align*}
\]

is $n_1 = n_1$ and $n_1 = n_1$.

**Proof.** [Proof of Corollary 1] Assume that $S$ is not minimal. Then there exists an equivalent $S_m$ of type $(n_{q'}, n_{q'})$ such that $n_{q'} + n_{q'} < n_{q'} + n_{q'}$. But this implies that $\dim \Sigma_{S_m} = n_{q'} + n_{q'} < n_{q'} + n_{q'} = \dim \Sigma_S$, which contradicts to the minimality of $\Sigma_S$.

**Proof.** [Proof of Corollary 2] Consider two SARX systems $S_1 = \{(n_q)_{q \in Q}\}$, $i = 1, 2$ of type $(n_{q}, n_{q})$. Notice that the associated DTLSSs $\Sigma_{S_1}$ and $\Sigma_{S_2}$ realize the same input-output map.

Assume that the parametrization $\Pi$ is identifiable, but $\Pi_{sw}$ is not identifiable. Then there exist two parameters $\theta_1, \theta_2 \in \Theta$, $\theta_1 \neq \theta_2$, such that $\Pi_{sw}(\theta_1)$ and $\Pi_{sw}(\theta_2)$ realize the same input-output map. Since $\Pi_{sw}(\theta_1) = \Sigma_{(\theta, j)}$, $i = 1, 2$, by the remark above it follows that $\Pi(\theta_1)$ and $\Pi(\theta_2)$ are equivalent. This contradicts the identifiability of $\Pi$.

Conversely, assume that $\Pi_{sw}$ is identifiable, but $\Pi$ is not identifiable. Then there exists parameters $\theta_1, \theta_2 \in \Theta$, $\theta_1 \neq \theta_2$, such that $\Pi(\theta_1)$ and $\Pi(\theta_2)$ are equivalent. This means that $\Sigma_{(\theta,j)} = \Sigma_{(\theta,j)}$ and $\Sigma_{(\theta,j)} = \Sigma_{(\theta,j)}$ realize the same input-output map. But this contradicts the identifiability of $\Pi_{sw}$.

For the proof of Theorem 1, we will need a number of auxiliary results. Below, we consider $A_S = \{(n_q)_{q \in Q}\}$. We denote by $A_q$ the corresponding matrix of the DTLSS $\Sigma_S$. We will by $e_i$ the $i$th standard basis vector of $\mathbb{R}^{n_{q}+n_{u}}$.

**Lemma 7.** Let $X_1 = \text{Span}\{e_1, \ldots, e_{n_q}\}$. It then follows that for any $q \in Q$,

1. $A_q e_j = n_q e_j + e_{j+1}$ for all $j = 1, \ldots, n_q + n_u$, $j \neq n_q$ and $A_q e_{n_q} = n_q e_{n_q}$.
2. The space $X_1$ is $A_q$ invariant and Span$\{(A_q e_1 | i = 0, \ldots, n_q - 1)\}$.
3. $A_q e_{n_q + j+1} = e_{n_q + j+1}$, $j = 0, \ldots, n_q - 1$, and $A_q e_{n_q + 1} = X_1$.
4. For any $q \in Q$, $\psi_{q,q,j}(A_q) e_j = A_q^2 e_j$, where $\psi_{q,q,0}(z) = 1$ and $\psi_{q,q,1}(z) = \theta - \theta z + (n_q - \theta) \phi_q, j = 0$.

**Proof.** [Proof of Lemma 7] Part 1 follows by a simple computation. Part 2 follows from Part 1 by taking into account that $A_q e_j \in \text{Span}\{e_1\}$. Part 3 follows from the definition of $A_q$ by induction. Indeed, $A_q e_{n_q + 1} = n_q e_{n_q + 1} + e_{n_q + 2}$ and $A_q e_{n_q + 1} = X_1 + e_{n_q + 1}$ and $A_q^2 e_{n_q + 1} = X_1 + e_{n_q + 1} \leq X_1 + e_{n_q + 1}$.

Finally, $\psi_{q,q,j}(A_q) e_j = A_q^2 e_j$ we will prove by induction. For $j = 0$, the equality is trivial. Notice that $A_q e_j = n_q e_j + e_{j+1} = A_q e_j + (n_q - n_q) e_j$ for all $j = 1, \ldots, n_q - 1$, and $A_q^2 e_j = n_q^2 e_j = A_q e_j + (n_q - n_q) e_j$. Hence, for any $x = \sum_{i=0}^{n_q} x_i e_i$, $A_q x = A_q x + \sum_{i=0}^{n_q} x_i (n_q - n_q) e_j = A_q x + ((n_q - n_q) x_i) e_i$. Hence, if $\psi_{q,q,j}(A_q) e_j = A_q^2 e_j$ holds, then $A_q^2 e_j = A_q \psi_{q,q,j}(A_q) e_j = A_q \psi_{q,q,j}(A_q) e_j + ((n_q - n_q) \psi_{q,q,j}(A_q) e_j) e_j$.

Finally, notice that $d_j = A_q e_j$ for all $j = 0, \ldots, n_q$. Indeed, $d_0 = e_1$, and if $d_j = \sum_{i=0}^{n_q} d_j e_i$, then $A_q d_j = \sum_{i=0}^{n_q} d_j e_i + \sum_{i=0}^{n_q} d_j e_j = d_{j+1}$. Hence, by replacing $\psi_{q,q,j}(A_q) e_j = A_q^2 e_j$ by $d_j$ in A.1, we obtain that $A_q^2 e_j = \psi_{q,q,j+1}(A_q) e_j$.

\footnote{Namely, the map which maps input $u_0, \ldots, u_t$ to the output $y_t = f((q_0, u_0) \cdots (q_t, u_t))$, where $f$ is the input-output map of $S$.}
Hence, by induction we get that the last statement of the lemma.

**Lemma 8.** If $n_q^{n_x+n_y} \neq 0$, then $\{e_{n_q}^{T} A_q^j | j = 0, \ldots, n_y + n_u - 1\}$ spans $\mathbb{R}^{x \times (n_x+n_u)}$.

Moreover, $e_{n_q}^{T} = e_{n_q}^{T} A_q^{-n_y}, i = 1, \ldots, n_y$, and $e_{n_q}^{T} A_q^j = e_{n_q}^{T} \chi_q(A_q)_{j,q}(A_q), j = 1, \ldots, n_u$, where $\chi_q(z) = z^{n_y} - \sum_{j=1}^{n_u} z^{n_y-j}$ and the polynomial $v_j(z), j = 1, \ldots, n_u$ is defined recursively as follows:

$$
\gamma_{1,q}(z) = \frac{1}{n_q^{n_x+n_y}} z^{n_y-1}
$$

$$
\gamma_{i,q}(z) = \frac{1}{n_q^{n_x+n_y}} (z^{n_y-i} - \sum_{j=1}^{i-1} \gamma_{j,q}(z) n_q^{n_y+n_u-i+j})
$$

**Proof.** [Proof of Lemma 9] From Lemma 8 it follows that $\mathbb{R}^{x \times (n_x+n_u)}$ is a cyclic subspace with respect to the linear operator $A_q : x^T \mapsto x^T A_q$. By (Gantmacher, 2000, Theorem 4, Chapter VII), it then follows that the minimal polynomial of the linear operator $A_q$ equals its characteristic polynomial and it is of degree $n_y + n_u$. Note that in the standard basis $e_1^T, \ldots, e_{n_q}^{T} A_q^j$, the basis of the linear operator $A_q$ is $A_q^j$. Hence, the minimal polynomial and characteristic polynomial of $A_q$ coincide. But these polynomials are the same for the matrices $A_q$ and $A_q^j$.

Moreover, from Lemma 8 it also follows that $e_{n_q}^{T}$ is the generating element of the cyclic space $\mathbb{R}^{x \times (n_x+n_u)}$. Hence, (by Gantmacher, 2000, §4.1, Chapter VII), a polynomial $\psi(z)$ is a minimal polynomial of $A_q$, if $\psi(A_q) e_{n_q}^{T}$ is reachable, and if Part (B) holds, then $\Sigma_S$ is observable.

**Proof of Part (A)**

We will show that if the conditions of (A) hold, then $(\Omega_S, A_q^n B_q)$ is a controllable pair. By Sun and Ge (2005a) it then follows that the DTLSS $\Sigma_S$ is reachable. From Lemma 7 it follows that $A_q^n B_q = A_q^n e_{n_q}^{T} A_q^{n_y+n_u}$ and hence

$$
\chi_q(z) = z^{n_y} - \sum_{j=1}^{n_u} z^{n_y-j-1}
$$

**Proof.** [Proof Theorem 1] We will show that if Part (A) holds, then $\Sigma_S$ is reachable, and if Part (B) holds, then $\Sigma_S$ is observable.

**Proof of Part (A)**

We will show that if the conditions of (A) hold, then $(\Omega_S, A_q^n B_q)$ is a controllable pair. By Sun and Ge (2005a) it then follows that the DTLSS $\Sigma_S$ is reachable. From Lemma 7 it follows that $A_q^n B_q = A_q^n e_{n_q}^{T} A_q^{n_y+n_u}$ and hence

$$
A_q^n B_q = \phi_{q_i,j}(A_q^n) e_{q_i,j+1}
$$

and hence the polynomial $\phi_{q_i,j}(z)$ satisfies

$$
A_q^n B_q = \phi_{q_i,j}(A_q^n) e_{q_i,j+1}
$$

see (Gantmacher, 2000, §4, Chapter VII) for the definition of cyclic subspaces.
From Lemma 7 it follows that \( \phi_{q_0,q_1}(A_{q_0})e_1 \in X_1 \) and \( X_1 \) is \( A_{q_0} \) invariant, where \( X = \text{Span}\{e_1, \ldots, e_{n_y}\} \). In addition, from the construction of \( A_{q_0} \) it follows that with respect to the basis \( e_1, \ldots, e_{n_y} \), the matrix representation of the restriction of \( A_{q_0} \) to \( X_1 \) is of the form

\[
\hat{A}_{q_0} = \begin{bmatrix}
[\begin{array}{c}
1_{n_y} \\
0
\end{array}] & \begin{bmatrix} n_{q_0}^1 & \cdots & n_{q_0}^{n_y-1} \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\begin{array}{c}
I_{n_y-1} \\
0_{n_y-1}
\end{array}
\end{bmatrix}
\begin{bmatrix}
0
\end{bmatrix}.
\]

The above matrix is in companion form and it is known that its characteristic polynomial of the linear operator \( \hat{A}_{q_0} \) restricted to \( X_1 \). Moreover, from Lemma 7 it follows that \( A_{q_0}^n e_1, j = 0, \ldots, n_y - 1 \) generate the space \( X_1 \). Then \( X_1 \) is a cyclic subspace w.r.t. \( A_{q_1} \).

Suppose now that \( \chi_{q_0}(z) \) and \( \phi_{q_0,q_1}(z) \) are coprime, but \( (A_{q_0}, A_{q_1}^{n_y}B_{q_1}) \neq (A_{q_0}, \phi_{q_0}(A_{q_0})e_1) \) is not a controllable pair. Then the vectors \( \begin{bmatrix} n_{q_0} \end{bmatrix} x, j = 0, \ldots, n_y - 1 \) satisfy \( \chi_{q_0}(z) = \phi_{q_0,q_1}(A_{q_0})e_1 \). But the degree of \( \chi_{q_0}(z) \) is especially small at most \( n_y - 1 \) such that \( \phi_{q_0,q_1}(A_{q_0})e_1 = 0 \). Hence, by the discussions above, \( \hat{A}_{q_0} \) is not invariant to \( X_1 \). We arrived to a contradiction. That is, we can conclude that \( (A_{q_0}, A_{q_1}^{n_y}B_{q_1}) \) is a controllable pair.

**Proof of (B)**

We will show that \( (C_{q_1}, A_{q_1}) \) is an observable pair. By Sun and Ge (2005a) this is sufficient for observability of \( \Sigma_S \).

To this end, using the notation of Lemma 8 define the polynomial

\[
\hat{\psi}(z) = v_{q_1}(z) + \sum_{j=1}^{n_y} n_{q_1}^{n_y+j} \gamma_{j,q_1}(z) \chi_{q_1}(z).
\]

Then from Lemma 8 it follows that \( C_{q_1} = e_{A_{q_1}}^T \hat{\psi}(A_{q_1}) \). Assume that \( (C_{q_1}, A_{q_1}) \) is not an observable pair. Then \( C_{q_1}A_{q_1}, j = 0, \ldots, n_y - 1 \) are linearly independent. Hence, there exists a polynomial \( \kappa(z) \) of degree less than \( n_y \), such that \( C_{q_1}\kappa(A_{q_1}) = 0 \). Hence, we obtain that \( e_{A_{q_1}}^T \hat{\psi}(A_{q_1})\kappa(A_{q_1}) = 0 \). In other words, the polynomial

\[
P(z) = \hat{\psi}(z)\kappa(z)\] is an annihilating polynomial with respect to the operator \( A_{q_2} : x \mapsto xA_{q_2} \) of \( e_{A_{q_1}}^T \). Since by Lemma 8 \( e_{A_{q_1}}^T A_{q_2}, j = 0, \ldots, n_y + n_u \) generate the whole space, it then follows that \( P(z) \) is the annihilating polynomial of the whole space, i.e. \( P(A_{q_2}) = 0 \). It then follows that \( P(z) \) is divisible by the minimal polynomial of \( A_{q_2} \), which coincides with that of \( A_{q_1} \). From Lemma 9 it follows that the minimal polynomial of \( A_{q_2} \) is \( z^{n_y} \chi_{q_2}(z) \). We will argue that if the conditions of Part (B) hold, then \( \hat{\psi}(z) \) and \( z^{n_y} \chi_{q_2}(z) \) are co-prime. Indeed, if \( \hat{\psi}(z) \) and \( z^{n_y} \chi_{q_2}(z) \) are not co-prime, then there exists an irreducible polynomial \( q(z) \) which divides both \( \hat{\psi}(z) \) and \( z^{n_y} \chi_{q_2}(z) \). If \( q(z) \) is an irreducible polynomial which divides \( z^{n_y} \chi_{q_2}(z) \), then it either equals \( z \) or it divides \( \chi_{q_2}(z) \). If \( q(z) = z \) and it divides \( \hat{\psi}(z) \), then \( 0 \) is a root of \( \hat{\psi}(z) \), i.e. \( \hat{\psi}(0) = 0 \). Notice that by induction it follows that for \( j = 1, \ldots, n_u - 1 \), \( \gamma_{j,q_2}(0) = 0 \) and \( n_{q_2}^{n_y-1} \). Hence, from the definition of \( \hat{\psi}(z) \) it follows that \( \hat{\psi}(0) = \frac{n_{q_2}^{n_y} + n_{q_2}^{n_y+n} + \cdots + n_{q_2}^{n_y+n_y}}{n_{q_2}^{n_y}} \chi_{q_2}(0) = \frac{n_{q_2}^{n_y+n} + \cdots + n_{q_2}^{n_y+n_y}}{n_{q_2}^{n_y}} \chi_{q_2}(0) \). Hence, \( \hat{\psi}(0) = 0 \) implies that \( n_{q_2}^{n_y} = \frac{n_{q_2}^{n_y+n} + \cdots + n_{q_2}^{n_y+n_y}}{n_{q_2}^{n_y}} \), which contradicts to the condition of (B). If \( q(z) \) divides \( \chi_{q_2}(z) \) and it divides \( \hat{\psi}(z) \), the it divides \( \chi_{q_2}(z) = \hat{\psi}(z) - (\sum_{i=1}^{n_y} n_{q_2}^{n_y+i} \gamma_{i,q_2}(z)) \chi_{q_2}(z) \). But this contradicts to the assumption that \( \psi_{q_1}(z) \) and \( \chi_{q_2}(z) \) are co-prime. Hence, by the above discussion, \( \hat{\psi}(z) \) and \( z^{n_y} \chi_{q_2}(z) \) are coprime, so if \( z^{n_y} \chi_{q_2}(z) \) divides \( P(z) \), it then must divide \( \kappa(z) \). But the degree of \( \kappa(z) \) is strictly smaller than that of \( z^{n_y} \chi_{q_2}(z) \), hence \( z^{n_y} \chi_{q_2}(z) \) cannot divide \( \kappa(z) \). We arrived to a contradiction. Hence, \( (C_{q_1}, A_{q_1}) \) must be an observable pair.

**Proof.** [Proof of Theorem 3] Assume that \( \Sigma_S = (p, m, n, Q, \{A_{q_1}, B_{q_1}, C_{q_1} \mid q \in Q \}) \). Then according to \( \Sigma_S \) and \( \Sigma_S' \), denote by \( e_i \) the ith standard unit vector of \( \mathbb{R}^n \). Then \( e_1, \ldots, e_n \) form the standard basis in \( \mathbb{R}^{1 \times n} \). The proof depends on the following series of technical results.

**Proposition 1.**

\[ e_i^T A_q = e_i^T M A_q \quad (A.6) \]

**Proof.** [Proof of Proposition 1] From the construction of \( \Sigma_S, i = 1, 2 \) it then follows that \( C_q = e_i^T A_q \), \( C_q' = e_i^T A_q' \). From the definition of isomorphism between DTLSSs, it follows that \( C_q' M = C_q, q \in Q \). Hence, we obtain that \( e_i^T A_q = e_i^T A_q' \). But \( A_q'M = M A_q \) by the definition of a DTLSS isomorphism, and hence we obtain the claim of the proposition.

**Proposition 2.** The columns of \( A_q \) span the space \( \text{Span}(e_1, \ldots, e_{n_u+n_u}) \).}

**Proof.** [Proof of Proposition 2] Indeed, \( A_q e_n = n_q^n e_1 \), \( A_q e_{n_u+n_u} = n_q^n e_1 \), \( A_q e_j = e_{j+1} + n_q^n e_1 \) for all \( j \in \{1, \ldots, n_u - 1 \} \). Hence, if either \( n_q^n \neq 0 \) or \( n_q^n \neq 0 \), then \( e_1 \) belongs to the column space of \( A_q \), and hence \( e_j = A_q e_{j-1} - n_q^{j-1} e_1 \) belongs to the column space of \( A_q \) for \( j \in \{2, \ldots, n_u \} \). Then for any \( i = 1, \ldots, n_u + n_y \), if \( e_i^T A_q = e_i^T M A_q \), then \( e_i^T = e_i^T M \).

**Proof.** [Proof of Proposition 3] Indeed, if \( e_i^T A_q = e_i^T M A_q \), then this implies that \( (e_i^T - e_i^T M) A_q = 0 \). By Proposition 2
this implies that $(e_j^T - e_j^T M) e_j = 0$ for all $j \in \{1, \ldots, n_u + n_y\} \setminus \{n_y + 1\}$. Notice that from the construction of $\Sigma_S$, $\Sigma_{S_2}$ and the definition of a DTLSS morphism it follows that $e_{n_y+1} = B_q = MB_q = Me_{n_y+1}$. Hence, $(e_j^T - e_j^T M)e_{n_y+1} = 0$ and thus

$$(e_j^T - e_j^T M)e_j = 0, j = 1, \ldots, n_y + n_u.$$  

This is just an alternative way of formulating the conclusion of the proposition.

**Proposition 4.** If $e_j^T M = e_j^T$, then $e_j^T A_q = e_j^T M A_q$ for all $j \in \{2, \ldots, n_y + n_u\} \setminus \{n_y + 1\}$. Hence, by using $A_q^T M = MA_q$, we derive

$$e_j^T M = e_j^T A_q^T M = e_j^T M A_q$$  

(A.7)

for all $j \in \{2, \ldots, n_y\} \cup \{n_y + 2, \ldots, n_y + n_u\}$. Since $e_j^T M = e_j^T$, and $e_j^T A_q = e_j^T$ for all $j \in \{2, \ldots, n_y + n_u\} \setminus \{n_y + 1\}$, from (A.7) we obtain the claim of the proposition.

The rest of the proof proceeds as follows. We will prove that

$$e_j^T = Me_j^T, j = 1, \ldots, n_y + n_u,$$  

(A.8)

which is just another way of saying that $M$ is the identity matrix. To this end, from (A.7) and Proposition 3 it follows that (A.8) holds for $j = 1$. Moreover, the $n_y + 1$th row of $A_q$ and $A_q$ are both zero, hence, $0 = e_{n_y+1}^T A_q = e_{n_y+1}^T A_q'$, and thus $0 = e_{n_y+1}^T A_q^T M = e_{n_y+1}^T M A_q$. From this we get that $e_{n_y+1}^T A_q = e_{n_y+1}^T M A_q$ and by Proposition 3 this implies that (A.8) holds for $j = n_y + 1$. Notice that if (A.8) holds for $j = k \in \{1, \ldots, n_y + n_y - 1\} \setminus \{n_y\}$, then by Proposition 4, $e_{k+1}^T A_q = e_{k+1}^T M A_q$. By Proposition 3, the latter implies that (A.8) holds for $j = k + 1$. Hence, by induction we get that (A.8) holds for all $j$.

**Proof.** [Proof of Theorem 2]  $e$ will show that the DTLSS parameterization $\Pi_{sw} : \theta \mapsto \Sigma_{\Pi(\theta)}$ is identifiable. By Corollary 2 this is sufficient for identifiability of $\Pi$.

Since $\Pi$ is strongly minimal, the DTLSS parameterization $\Pi_{sw}$ is minimal, see Petreczky et al. (2010) for definition. In order to show identifiability of $\Pi_{sw}$, by (Petreczky et al., 2010, Corollary 1) it is enough to show that the only isomorphism between elements of $\Pi_{sw}$ is the identity. Consider now two elements $\Sigma_i = \Sigma_{\Pi(\theta_i)}$, $\theta_i \in \Theta$, $i = 1, 2$ of $\Pi_{sw}$. Notice that $\Pi(\theta_i)$ is minimal, since it is strongly minimal. If $\Pi(\theta_i)$ is a strongly minimal element, then we have $\Sigma_i = \Sigma_{\Pi(\theta_i)}$, $\theta_i \in \Theta$, $i = 1, 2$ of $\Pi_{sw}$. Notice that $\Pi(\theta_i)$ is minimal, since it is strongly minimal. If $\Pi(\theta_i)$ is a strongly minimal element, then we have $\Sigma_i = \Sigma_{\Pi(\theta_i)}$, $\theta_i \in \Theta$, $i = 1, 2$ of $\Pi_{sw}$.

**Proof.** [Proof of Proposition 4] Let $K = \{m_{n_q+n_u}Q\}$. Then any SARX system of type $(n_q, n_u)$ can be identified with a point in $R^K$, by identifying the system with the collection of its parameters $\{n_q\}_{q \in Q}$, $n_q \in R^{p \times (m_{n_q+n_u})}$.

First, we construct a polynomial $P_{obs}(X_1, \ldots, X_K)$, such that $P_{obs}(S) \neq 0$ if and only if $S$ is strongly minimal. To this end, consider the DTLSS $\Sigma_S$ and consider the observability and controllability matrix $O(\Sigma_S)$ and $R(\Sigma_S)$.

Any $n \times n$ minor of $O(\Sigma_S)$ or of $R(\Sigma_S)$ can be viewed as a polynomial in the entries of the matrices of $\Sigma_S$. Since the entries of the matrices of $\Sigma_S$ are linear functions of the entries of the parameters of $S$, it follows that any $n \times n$ minor of $O(\Sigma_S)$ and $R(\Sigma_S)$ can be viewed as polynomial in $S$, where $S$ is identified with an element of $R^K$. Let $P_1$ and $P_2$ be the set of all $n \times n$ minors of $O(\Sigma_S)$ and respectively $R(\Sigma_S)$, each minor being viewed as a polynomial in $R^K$. Define the polynomials

$$P_{obs}(X_1, \ldots, X_K) = \sum_{p \in P_1} (P(X_1, \ldots, X_K))^2$$

$$P_{cont}(X_1, \ldots, X_K) = \sum_{p \in P_2} (P(X_1, \ldots, X_K))^2$$

It is then clear that $P_{obs}(S) \neq 0$ if and only if at least one of the $n \times n$ minors of $O(\Sigma_S)$ is not zero, i.e. if and only if $\text{rank}(O(\Sigma_S)) = n$. That is, $P_{obs}(S) \neq 0$ if and only if $\Sigma_S$ is observable. Similarly, $P_{cont}(S) \neq 0$ if and only if $\text{rank}(R(\Sigma_S)) = n$ if and only if $\Sigma_S$ is reachable. Define now $P_{min} = P_{obs} P_{cont}$. Then $P_{min}(S) \neq 0$ if and only if $\Sigma_S$ is both observable and reachable, i.e. if and only if $\Sigma_S$ is minimal.

Finally, consider a polynomial parametrization $\Pi$ such that $\Pi$ contains a strongly minimal element. The fact that $\Pi$ is a polynomial parametrization implies that there exists polynomials $\Pi$, in variables $X_1, \ldots, X_d$, $i = 1, \ldots, K$ such that $\Pi(\theta) = (\Pi_1(\theta), \ldots, \Pi_K(\theta))$ for all $\theta \in \Theta$. Here we used the identification of a SARX system of type $(n_q, n_u)$ with a point in $R^K$. Consider the polynomial

$$Q_{min}(X_1, \ldots, X_d) = P_{min}(\Pi_1(X_1, \ldots, X_d), \ldots, \Pi_K(X_1, \ldots, X_d)).$$

Notice that the set of parameters from $\Theta$ which do not yield a minimal SARX system all satisfy the equation $Q_{min}(\theta) = 0$. From the assumption that $\Pi$ contains a strongly minimal element it follows that for some $\theta \in \Theta$, $Q_{min}(\theta) = P_{min}(\Pi(\theta)) \neq 0$. Hence, the set $G = \{\theta \in \Theta \mid Q_{min}(\theta) = 0\}$ is a non-empty subset of $\Theta$ and it is clearly generic. That is, $\Pi$ is generically strongly minimal, and hence minimal.

**Proof.** [Proof of Corollary 3] If $\Pi$ is generically strongly minimal, then there exists a generic set $G \subseteq \Theta$ such that the parametrization $\Pi_{|G} : G \mapsto \Pi(\theta)$ is strongly minimal. Hence, by Theorem 2, $\Pi_{|G}$ is identifiable. This means that $\Pi$ is generically strongly identifiable.

**Proof.** [Proof of Corollary 4] If $\Pi$ contains a strongly minimal element, then by Theorem 4 $\Pi$ is generically strongly minimal. The rest follows from Corollary 3.

**Proof.** [Proof of Corollary 5] By Example 2, there exists a strongly minimal SARX system, i.e. $\text{SARX}_{triv}$ contains a strongly minimal element. Moreover, $\text{SARX}_{triv}$ is clearly injective and polynomial. The statement follows now Theorem 4 and Corollary 4.